LEARNING GUIDE

MATHEMATICS DBE

SOLUTIONS

MTH-5173-2 SCI

VOLUME 1

GEOMETRIC REPRESENTATION IN A FUNDAMENTAL CONTEXT 2

IN COMPLIANCE WITH THE NEW PROGRAM OF STUDY

SOFAD

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VOLUME 1

GEOMETRIC REPRESENTATION

IN A FUNDAMENTAL CONTEXT 2

SOFAD

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Legend: r = right c = centre l = left
t = top b = bottom

Table of Contents VOLUME 1

This preview contains:
- The Table of Contents;
- The Introduction;
- The First Situation.

CHAPTER 3

COMPLEMENTS

HOW THE LEARNING GUIDE IS STRUCTURED

Welcome to the learning guide for the ***Geometric Representation in a Fundamental Context 2*** course. The aim of this course, which is the third in the Secondary V ***Science*** sequence, is to develop your skills in dealing with situations based on descriptions or spatial representations. To achieve this, you will study geometric concepts, namely:

- the circle
- the ellipse
- the hyperbola
- the parabola

You will discover the concept of vectors and their properties for the first time.

You will complete your learning by expanding your knowledge of:

- trigonometric identities
- equivalent figures
- geometric transformations

You will be required to use various solution strategies to understand and model situational problems. You will need to use your mathematical reasoning skills. You will also have to describe how you solved these problems clearly and thoroughly using mathematical language. You are now invited to complete the learning activities found in the five chapters of the two guides for this course and enrich your knowledge of geometry.

Portailsofad.com

Go to portailsofad.com for videos, ICT activities and printable versions of resources that are complementary to the SOLUTIONS series, which you can use throughout your learning journey.

CHAPTER COMPONENTS

The learning process followed in each chapter enables students to progress by building on what they have learned from one section to the next. The following diagrams illustrate this approach and specify the pedagogical intent of each section.

CHAPTER INTRODUCTION

The first page describes the context and theme that will serve as a backdrop for the acquisition of the new knowledge discussed in the chapter.


CHAPTER 1

Trigonometric Relations

Designing Spaces and Objects

To design spaces or objects that meet restrictions, it is often necessary to use mathematical reasoning. In some situations, trigonometric calculations must be used to determine precise length or angle measures. For example, to respond to the request of a jeweller who wants to calculate the quantity of precious metal certain pendants must contain. Or to determine the precise position of the lines that must be drawn to construct a sundial. Or you may even call on trigonometric relations to determine the minimum width a building corridor must have so that furniture can be transported easily.

Trigonometric relations have already been discussed in a previous geometry course. The next chapter will deal with the meaning of the sine and the cosine of an angle. This time, the study of these relations will not just be limited to right triangles: trigonometry will be presented in the context of a circle, as the mathematician-astronomers of Antiquity and the Middle Ages initially approached it.

SITUATION 1.1
THE SINE AND THE COSINE
THE SUM AND DIFFERENCE OF ANGLES (OF THE SINE AND COSINE)
THE TRIGONOMETRIC UNIT CIRCLE (RADIAN AND ARC LENGTH)
SP 1.1 - A Star Pendant p. 4

SITUATION 1.2
THE TANGENT
THE TRIGONOMETRIC RATIOS SECANT, COSECANT AND COTANGENT
THE SUM AND DIFFERENCE OF ANGLES (OF THE TANGENT)
TRIGONOMETRIC IDENTITIES
SP 1.2 - Constructing a Sundial p. 38

KNOWLEDGE SUMMARY p. 70
INTEGRATION p. 78
LES
A Sofa Problem p. 86

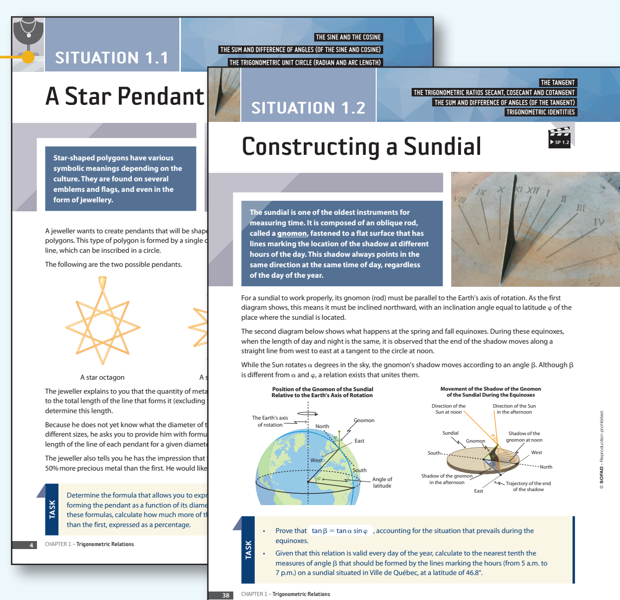
CHAPTER 1

2 CHAPTER 1 - Trigonometric Relations 3

A table of contents accompanies this first page. The knowledge to be acquired is described for each of the *Situations*, as well as the theme of the situational problems.

SITUATIONS

In general, there are two learning *Situations* per chapter. The approach taken in these situations makes it possible to acquire new knowledge and develop mathematical skills in real, realistic or purely mathematical contexts.


SITUATION 1.1

A Star Pendant

THE SINE AND THE COSINE
THE SUM AND DIFFERENCE OF ANGLES (OF THE SINE AND COSINE)
THE TRIGONOMETRIC UNIT CIRCLE (RADIAN AND ARC LENGTH)

Star-shaped polygons have various symbolic meanings depending on the culture. They are found on several emblems and flags, and even in the form of jewellery.

SITUATION 1.2

Constructing a Sundial

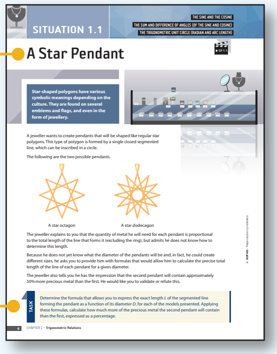
THE TANGENT
THE TRIGONOMETRIC RATIOS SECANT, COSECANT AND COTANGENT
THE SUM AND DIFFERENCE OF ANGLES (OF THE TANGENT)
TRIGONOMETRIC IDENTITIES

The sundial is one of the oldest instruments for measuring time. It is composed of an oblique rod, called a gnomon, fastened to a flat surface that has lines marking the location of the shadow at different hours of the day. This shadow always points in the same direction at the same time of day, regardless of the day of the year.

TASK

38 CHAPTER 1 - Trigonometric Relations

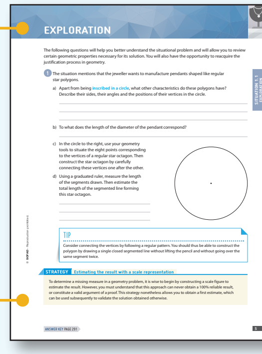
PHASES OF EACH SITUATION



SITUATIONAL PROBLEM

Linked to the main theme of the chapter, this page briefly describes the context of the situational problem, as well as the information required to solve it.

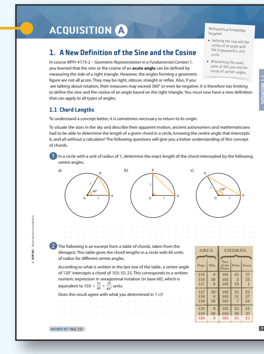
A box describes the task you will have to perform later in the *Solution* section. This task is the starting point for acquiring new knowledge to solve the situational problem.



EXPLORATION

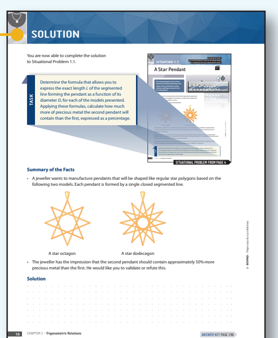
This section invites you to analyze the data of a situational problem, and then to identify the knowledge that you possess and the knowledge you need to acquire in order to perform the task.

The questions posed will guide you toward a problem-solving strategy.



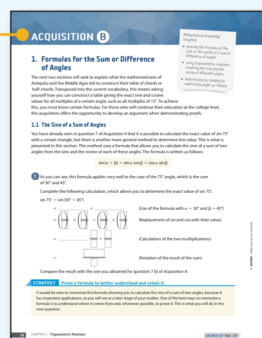
ACQUISITION A

This is where the knowledge needed to solve the situational problem is assimilated. Each *Acquisition* encourages reflection before presenting new mathematical knowledge.



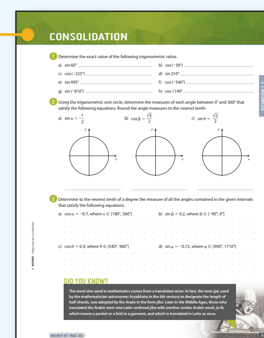
SOLUTION

By the time you reach this section, you should have acquired all the knowledge and strategies that are essential to solving the situational problem described at the beginning of the situation.



ACQUISITION B

In this second acquisition, you will acquire new knowledge prescribed by the program linked to the knowledge encountered in *Acquisition A*.



CONSOLIDATION

This section will allow you to consolidate the mathematical knowledge acquired in *Acquisitions A* and *B*. As in the *Integration* section, this *Consolidation* also contributes to the development of mathematical skills.

AT THE END OF A CHAPTER...

KNOWLEDGE SUMMARY

This section summarizes all the knowledge to *Remember* in the form of fill-in-the-blank questions. We invite you to fill in the missing information.

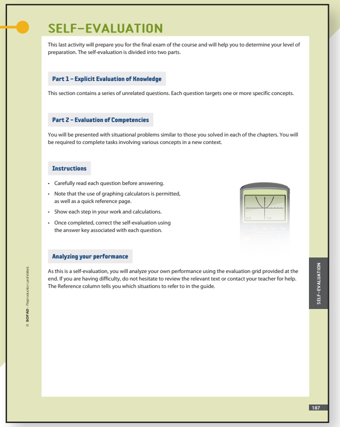
INTEGRATION

In this section, which includes exercises and complex situations, you will have to apply the knowledge seen in this chapter.

LES

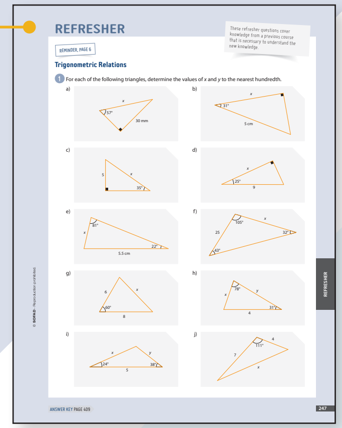
The *LES* is a complex task developed according to the certification evaluation model. It is accompanied by a competency evaluation grid.

COMPLEMENTS



SELF-EVALUATION

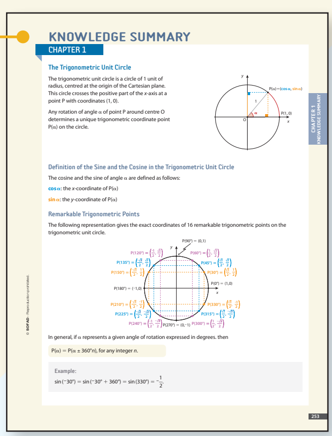
A *Self-evaluation* is presented in the first part of the Complements in Volume 2. It allows you to evaluate your acquired knowledge and the mathematical skills you have developed throughout the course. In this way, you will be able to identify the knowledge that you have mastered and that for which a revision is necessary before moving on to the *Summary Scored Activity*.



REFRESHER

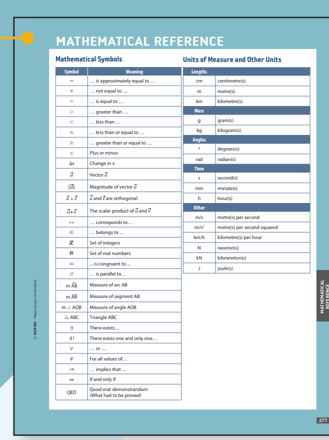
Throughout the *Situations*, you will come across headings entitled *Reminder*. These sections present concepts seen in a previous course that are necessary to understand the new knowledge or to solve the current situation.

The *Refresher* section allows you to use exercises to review the mathematical rules and concepts that are the subject of a *Reminder*.



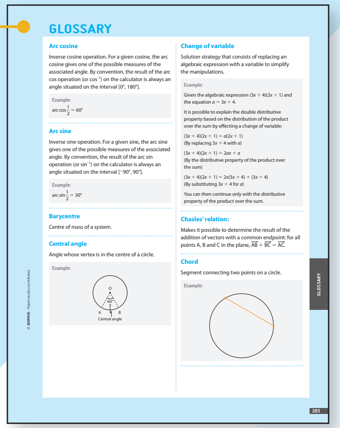
KNOWLEDGE SUMMARY

The full version of the *Knowledge Summary* is found in this section. A printable version is also available online.



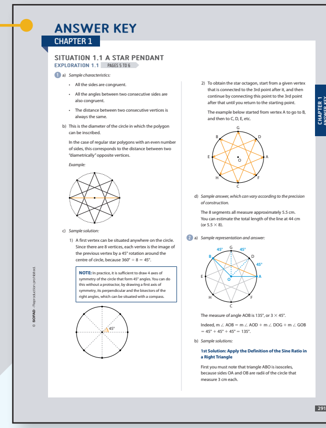
MATHEMATICAL REFERENCE

In this section, we present mathematical symbols used in the guide and some abbreviations of units of measurement. Reminders of mathematical formulas are also provided.



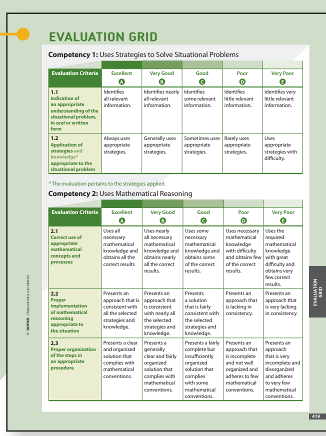
GLOSSARY

Words and expressions **written in blue** in the current text are defined in the *Glossary*.



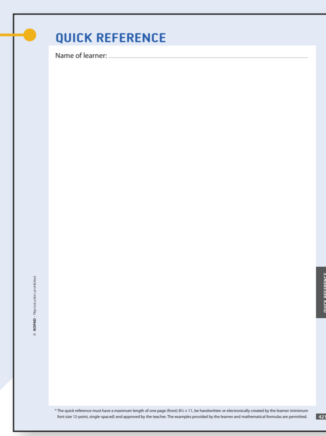
ANSWER KEY

Toward the end of the guide, you will find the *Answer Key*. It is designed not only for checking your answers, but also to complement your learning process. It contains the answers to questions and detailed explanations of the approach to be taken or the reasoning to be used.



EVALUATION GRID

A competency *Evaluation Grid* is available at the end of the guide. After solving an *LES*, you are asked to evaluate yourself using this grid. You can then complete the abbreviated version at the bottom of each *LES*.



QUICK REFERENCE

You can create your own quick reference guide. A detachable sheet is provided for this purpose at the end of the guide. You may use this quick reference during the final test.

HEADINGS AND PICTOGRAMS

Invites the student to watch a video clip on the situational problem.

TASK

Determine the formula that allows you to express the exact length *L*…

Presents the task to be performed as part of your Situational Problem.

REMINDER

REFRESHER EXERCISES
PAGE 249, QUESTIONS 6 TO 7

Solving…

The quadratic formula…

Example:

Solve the equation…

Refers to knowledge you have acquired in previous courses and refresher exercises related to this *Reminder*.

REMEMBER

Definition of the Sine…

The **trigonometric unit circle** …

Example:

The **sine law** allows you…

Presents the mathematical knowledge you will be required to master. This is the knowledge prescribed by the study program.

STRATEGY **Estimate a…**

To determine a missing measure in a geometry problem,…

Presents problem-solving strategies that can be applied to a variety of situations.

DID YOU KNOW?

Although the number 3438 may seem strange, Aryabhata referred to it, …

Allows you to discover historical and cultural information related to the mathematical concepts being studied.

TIP

It is important to note that this definition of the sine and the cosine in a circle of radius 1 does not contradict what you…

Provides a tip that simplifies the task, or offers a different way of dealing with the problem or of applying the concept being studied.

CAUTION!

This first chapter concerning the concepts involved in trigonometric relations and proofs is the most difficult…

Warns of traps to avoid or exceptions that may apply to the concept being studied.

ICT

In ICT activity 2.1.1, you can observe composites of translation sequences using GeoGebra. Find the activity on portailsofad.com…

Prompts you to complete an online activity (GeoGebra or graphing calculator) that will encourage you to explore the concept studied using technological tools.

SCORED ACTIVITY

You must now complete Scored Activity 1. It can be found on the course website…

Indicates that you are ready to complete the *Scored Activity* designed to assess your comprehension as you learn. The *Summary Scored Activity* is completed at the very end of the course. These activities are presented in separate booklets of the guide. You will have to submit each completed activity to your teacher or tutor who will provide you with feedback following correction.

Trigonometric Relations

Designing Spaces and Objects

To design spaces or objects that meet restrictions, it is often necessary to use mathematical reasoning. In some situations, trigonometric calculations must be used to determine precise length or angle measures. For example, to respond to the request of a jeweller who wants to calculate the quantity of precious metal certain pendants must contain. Or to determine the precise position of the lines that must be drawn to construct a sundial. Or you may even call on trigonometric relations to determine the minimum width a building corridor must have so that furniture can be transported easily.

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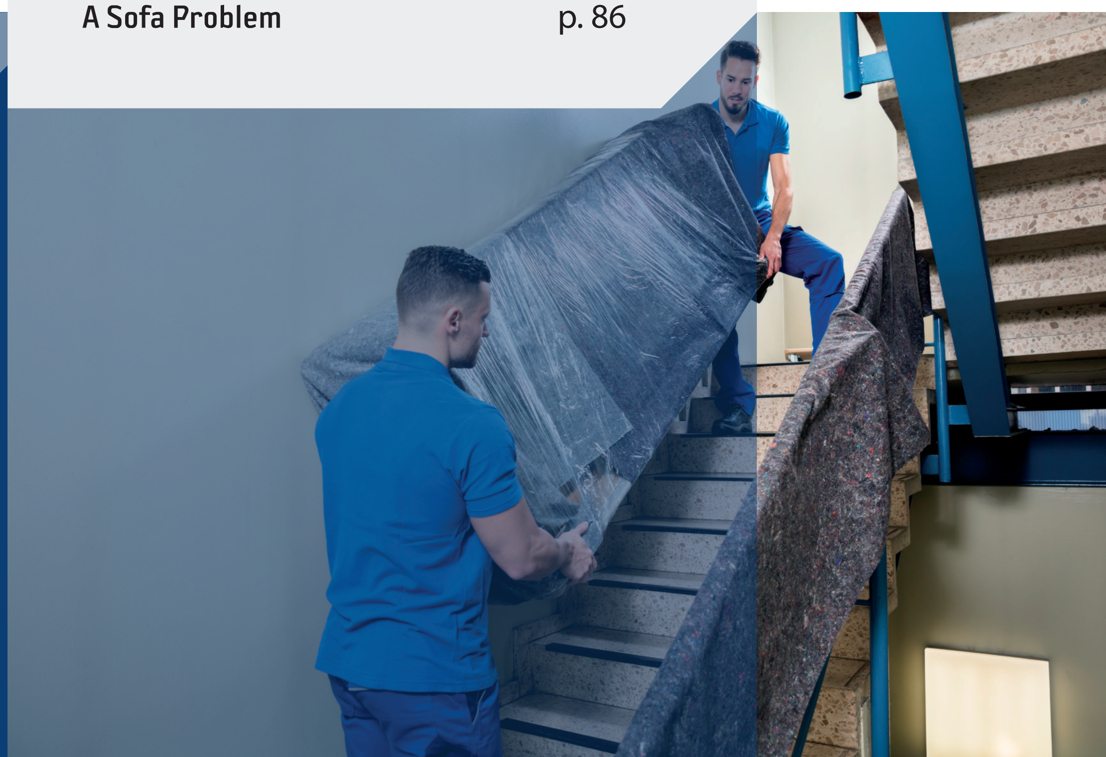
CHAPTER 1



SITUATION 1.1

THE SINE AND THE COSINE

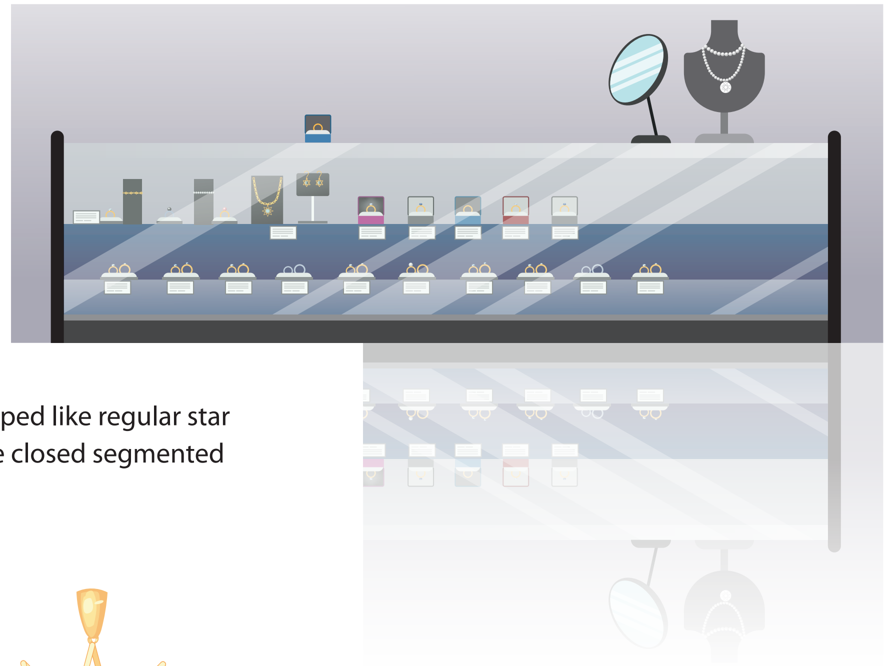
THE SUM AND DIFFERENCE OF ANGLES (OF THE SINE AND COSINE)

THE TRIGONOMETRIC UNIT CIRCLE (RADIAN AND ARC LENGTH)

SP 1.1

A Star Pendant

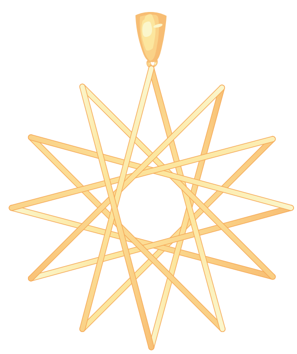
Star-shaped polygons have various symbolic meanings depending on the culture. They are found on several emblems and flags, and even in the form of jewellery.



A jeweller wants to create pendants that will be shaped like regular star polygons. This type of polygon is formed by a single closed segmented line, which can be inscribed in a circle.

The following are the two possible pendants.

A star octagon



A star dodecagon

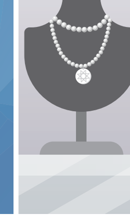
The jeweller explains to you that the quantity of metal he will need for each pendant is proportional to the total length of the line that forms it (excluding the ring), but admits he does not know how to determine this length.

Because he does not yet know what the diameter of the pendants will be and, in fact, he could create different sizes, he asks you to provide him with formulas that would allow him to calculate the precise total length of the line of each pendant for a given diameter.

The jeweller also tells you he has the impression that the second pendant will contain approximately 50% more precious metal than the first. He would like you to validate or refute this.

TASK

Determine the formula that allows you to express the exact length *L* of the segmented line forming the pendant as a function of its diameter *D*, for each of the models presented. Applying these formulas, calculate how much more of the precious metal the second pendant will contain than the first, expressed as a percentage.



EXPLORATION

The following questions will help you better understand the situational problem and will allow you to review certain geometric properties necessary for its solution. You will also have the opportunity to reacquire the justification process in geometry.

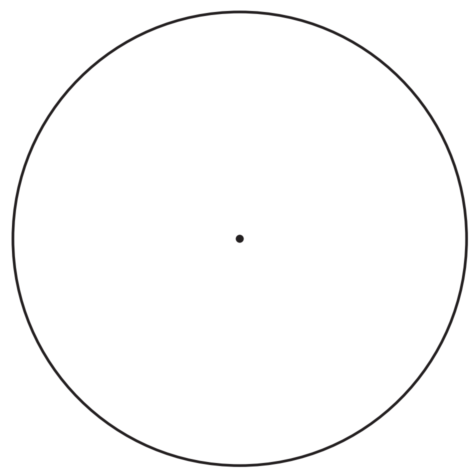
1 The situation mentions that the jeweller wants to manufacture pendants shaped like regular star polygons.

a) Apart from being **inscribed in a circle**, what other characteristics do these polygons have? Describe their sides, their angles and the positions of their vertices in the circle.

b) To what does the length of the diameter of the pendant correspond?

c) In the circle to the right, use your geometry tools to situate the eight points corresponding to the vertices of a regular star octagon. Then construct the star octagon by carefully connecting these vertices one after the other.

d) Using a graduated ruler, measure the length of the segments drawn. Then estimate the total length of the segmented line forming this star octagon.



TIP

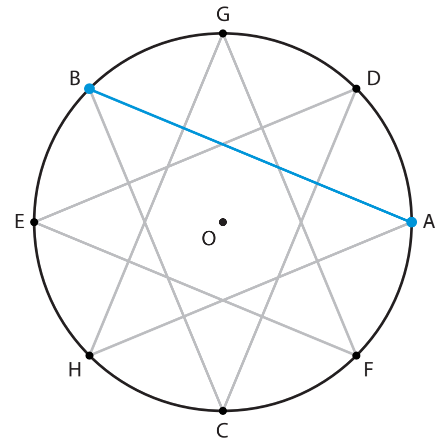
Consider connecting the vertices by following a regular pattern. You should thus be able to construct the polygon by drawing a single closed segmented line without lifting the pencil and without going over the same segment twice.

STRATEGY Estimating the result with a scale representation

To determine a missing measure in a geometry problem, it is wise to begin by constructing a scale figure to estimate the result. However, you must understand that this approach can never obtain a 100% reliable result, or constitute a valid argument of a proof. This strategy nonetheless allows you to obtain a first estimate, which can be used subsequently to validate the solution obtained otherwise.

ANSWER KEY PAGE 291

2 Below is a slightly smaller representation of the figure constructed in question 1. Each side of the star octagon ABCDEFGH is a **chord** of the circle. Chord AB has been highlighted.



a) Draw the **centre angle** AOB, which intercepts this chord. What is the measure of this angle? Justify your answer.

b) The centre angle and the chord form a triangle ABO. Assuming that the diameter of the circle is 6 cm, determine the measure of side AB of this triangle to the nearest tenth of a millimetre.

c) Estimate the total length of the segmented line forming the star octagon to the nearest millimetre.

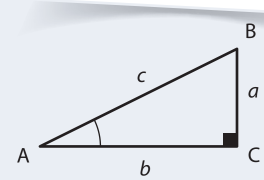
REMINDER

REFRESHER EXERCISES
PAGES 247 AND 248, QUESTIONS 1 TO 3

Trigonometric Relations

Trigonometric Ratios

In a **right triangle**, you can calculate an unknown measure from two known measures of this triangle by using **trigonometric ratios**.



$$\sin A = \frac{\text{Measure of the side opposite to angle A}}{\text{Measure of the hypotenuse}} = \frac{a}{c}$$

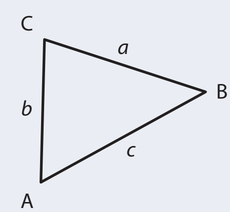
$$\cos A = \frac{\text{Measure of the side adjacent to angle A}}{\text{Measure of the hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{Mesure of the side opposite to angle A}}{\text{Measure of the side adjacent to angle A}} = \frac{a}{b}$$

Sine Law

The sides of a triangle are proportional to the sine of the opposite angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Cosine Law

The square of the length of a side of a **given triangle** is equal to the sum of the squares of the lengths of the other sides, minus double the product of the lengths of the other sides times the cosine of the angle between these two sides.

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad b^2 = a^2 + c^2 - 2ac\cos B \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

In this *Exploration*, you estimated the length of the segmented line forming a star octagon with a certain diameter. However, you must understand that this approximation is not enough to solve the situational problem, because you are looking for formulas that will determine the precise lengths of the pendants. To determine the parameters of these formulas, you must know more about the **sine** and the **cosine** of an angle. This is what is presented in *Acquisition A*.

ANSWER KEY PAGE 291

ACQUISITION A

Mathematical knowledge targeted:

- defining the sine and the cosine of an angle with the trigonometric unit circle
- determining the exact value of the sine and the cosine of certain angles.

1. A New Definition of the Sine and the Cosine

In course MTH-4173-2 – *Geometric Representation in a Fundamental Context 1*, you learned that the sine or the cosine of an **acute angle** can be defined by measuring the side of a right triangle. However, the angles forming a geometric figure are not all acute. They may be right, obtuse, straight or reflex. Also, if you are talking about rotation, their measures may exceed 360° or even be negative. It is therefore too limiting to define the sine and the cosine of an angle based on the right triangle. You must now have a new definition that can apply to all types of angles.

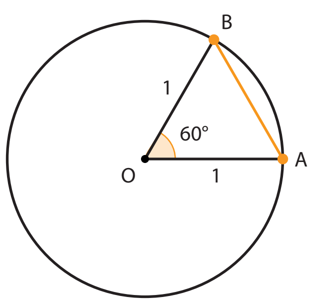
1.1 Chord Lengths

To understand a concept better, it is sometimes necessary to return to its origin.

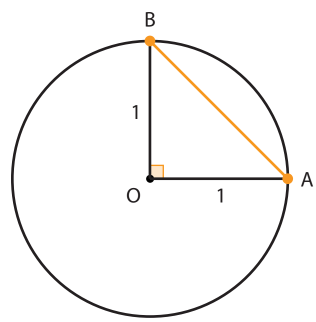
To situate the stars in the sky and describe their apparent motion, ancient astronomers and mathematicians had to be able to determine the length of a given chord in a circle, knowing the centre angle that intercepts it, and all without a calculator! The following questions will give you a better understanding of this concept of chords.

1 In a circle with a unit of radius of 1, determine the exact length of the chord intercepted by the following centre angles.

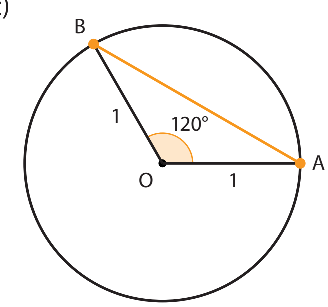
a)



b)



c)



2 The following is an excerpt from a table of chords, taken from the *Almagest*. This table gives the chord lengths in a circle with 60 units of radius for different centre angles.

According to what is written in the last row of the table, a centre angle of 120° intercepts a chord of 103; 55; 23. This corresponds to a written numeric expression in sexagesimal notation (in base 60), which is equivalent to $103 + \frac{55}{60} + \frac{23}{60^2}$ units.

Does this result agree with what you determined in 1 c)?

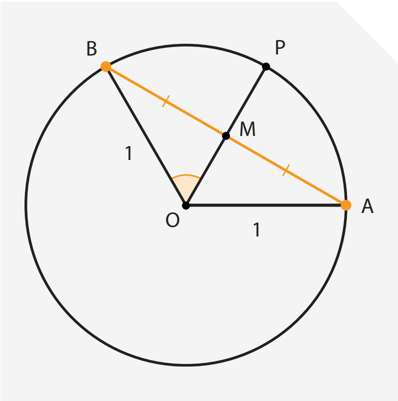
ARCS.		CHORDS.		
Degr.	Min.	Part of the Diam.	Prim.	Secon.
116	0	101	45	57
116	30	102	2	33
117	0	102	19	1
117	30	102	35	22
118	0	102	51	37
118	30	103	7	44
119	0	103	23	44
119	30	103	39	37
120	0	103	55	23

ANSWER KEY PAGE 293

1.2 From the Chord to the Half Chord

The next part of this story took place in India in the 6th century. The mathematician-astronomer Aryabhata was interested not only in the length of chords, but the length of half-chords.

3 To make a connection with the previous discussion, return to the figure from question 1 c), in which the circle has 1 unit of radius and angle AOB measures 120°. Point M has been situated at the midpoint of chord AB and radius OP has been drawn, passing through M.



a) What is the measure of angle BOM? ______

b) What is the exact length of segment BM? ______

c) Determine what the exact length of half-chord BM would be:

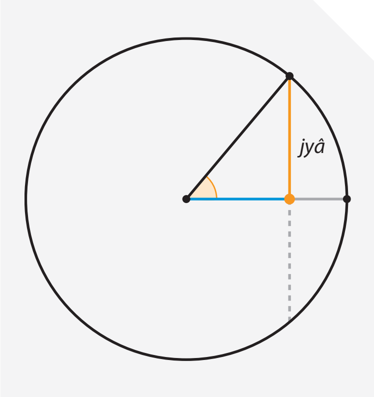
1) if angle BOM measured 45°: ______

2) if angle BOM measured 30°: ______

4 The following table of values gives some half-chord lengths calculated by Aryabhata for a circle with 3438 units of radius. The Hindi word *jyâ*, used by Aryabhata, designates this half-chord length.

Half-Chord Length According to the Centre Angle Measure

Angle (degrees)	*jyâ*	*jyâ* ÷ 3438
0°	0	
15°	890	
30°	1719	
45°	2431	
60°	2978	
75°	3321	
90°	3438	



a) Complete the last column of the table by dividing the length of the half-chord (*jyâ*) by the radius of the circle (3438) to obtain the lengths of the half-chords in a circle with 1 unit of radius Round to the nearest four decimal places.

b) Compare the values of this third column with the sine of each angle the calculator gives you. What do you observe?

______

DID YOU KNOW?

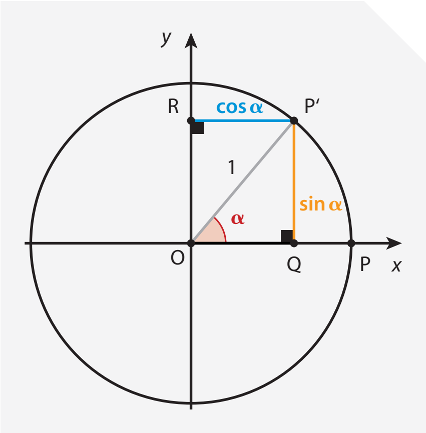
Although the number 3438 may seem strange, Aryabhata referred to it, because his table used a circle with a circumference that measured 21 600 units, or 60 × 360, to reference degrees and have a practical use of base 60. Thus, with a radius of 3438 units, π corresponds approximately to 3.14136.

ANSWER KEY PAGE 293

1.3 From the Half-Chord to the Sine and the Cosine of an Angle

What you must remember is that the sine of an angle is essentially the length of a half-chord, as Aryabhata defined it, but in a circle with 1 unit of radius. There is one last step to take: discover the current definition of the sine and the cosine. To do this, the circle must be placed in a Cartesian plane.

5 In a circle with 1 unit of radius centred at the origin of the Cartesian plane, point P has been situated on the *x*-axis with coordinates (1, 0). Point P′ is the image of point P by rotation of an acute angle α around O. The figure has been completed by drawing rectangle OQP′R.



In accordance with what you have seen previously, **sin α** can be defined as the measure of half- chord **QP′**. You can also define **cos α** as the measure of the half-chord intercepted by the complementary angle, half-chord **RP′**.

a) What trigonometric expressions represent the measure of the other sides of rectangle OQP′R?

b) What expressions represent the coordinates of point P′?

c) What relation involving $\sin\alpha$ and $\cos\alpha$ is obtained if the Pythagorean theorem is applied to triangle OP′Q?

d) In question 3, you calculated the length of the half-chords intercepted by the 30°, 45° and 60° centre angles. In other words, you determined the following exact values:

$\sin 30° = \frac{1}{2}$ $\sin 45° = \frac{\sqrt{2}}{2}$ $\sin 60° = \frac{\sqrt{3}}{2}$

Given your answers in a) and c) of this question, deduce the exact values of cos 30°, cos 45° and cos 60°.

TIP

It is important to note that this definition of the sine and the cosine in a circle of radius 1 does not contradict what you learned in the previous geometry course. In a right triangle, the sine of an acute angle always corresponds to the measure of the opposite side divided by the measure of the hypotenuse, and the cosine always corresponds to the measure of the adjacent side divided by the measure of the hypotenuse. This is also the case in right triangle OP′Q.

$$\sin\alpha = \frac{m\,\overline{QP'}}{m\,\overline{OP'}} = \frac{m\,\overline{QP'}}{1} = m\,\overline{QP'} \qquad \cos\alpha = \frac{m\,\overline{OQ}}{m\,\overline{OP'}} = \frac{m\,\overline{OQ}}{1} = m\,\overline{OQ}$$

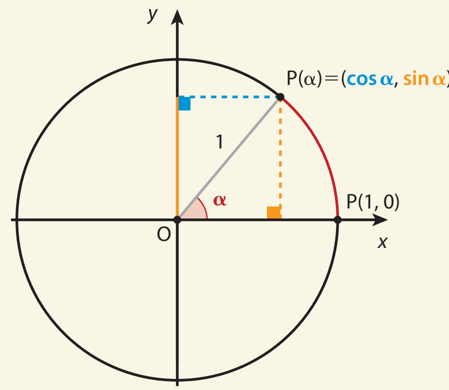
As you will soon recognize, the advantage of the new definition is that it can be generalized for all types of angles.

ANSWER KEY PAGE 293

REMEMBER

The Definition of Sine and Cosine in the Trigonometric Unit Circle

The **trigonometric unit circle** is a circle of 1 unit of radius, centred at the origin of the Cartesian plane. This circle crosses the positive portion of the x-axis at a point P of coordinates (1, 0).



Any rotation of angle α of point P(1, 0) around centre O determines a unique **trigonometric point** of coordinates P(α) on the circle.

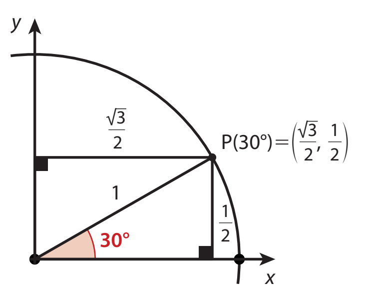
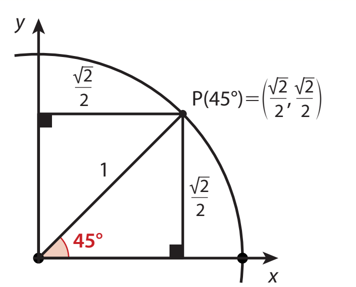
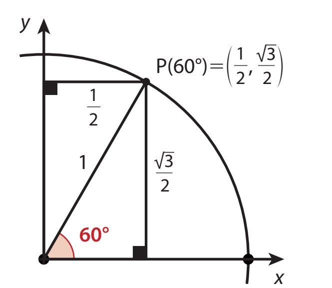
The cosine and the sine of angle α are defined as follows:

cos α: the x-coordinate of P(α)

sin α: the y-coordinate of P(α)

The Exact Values of the Sine and Cosine of the 30°, 45° and 60° Angles

You can determine the exact values of the sine and cosine of the 30°, 45° and 60° angles.

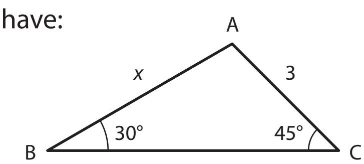
30°	45°	60°
		
$\cos 30° = \frac{\sqrt{3}}{2}$ $\sin 30° = \frac{1}{2}$	$\cos 45° = \frac{\sqrt{2}}{2}$ $\sin 45° = \frac{\sqrt{2}}{2}$	$\cos 60° = \frac{1}{2}$ $\sin 60° = \frac{\sqrt{3}}{2}$

Knowing these values allows you to determine the exact values of missing measures in certain geometric figures.

Example:

The **sine law** allows you to determine the measure of side AB in the triangle below.

Indeed, according to this property, $\frac{m\,\overline{AB}}{\sin C} = \frac{m\,\overline{AC}}{\sin B}$. You therefore have:



$$\frac{x}{\sin 45°} = \frac{3}{\sin 30°}$$

$$x = \frac{3 \sin 45°}{\sin 30°}$$

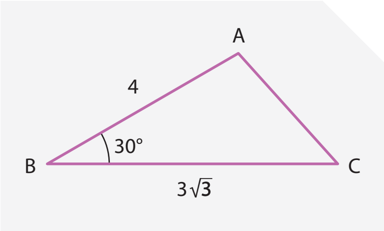
$$x = \frac{3 \times \left(\frac{\sqrt{2}}{2}\right)}{\frac{1}{2}}$$

$$x = 3\sqrt{2}$$

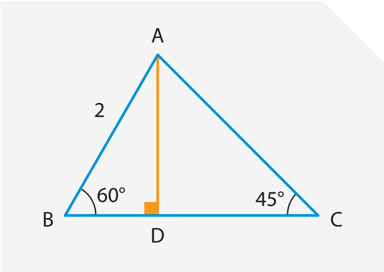
The exact measure of side AB is $3\sqrt{2}$ units.

PRACTISE

6 Applying the cosine law, determine the exact measure of side AC in the triangle below.



7 Altitude AD in triangle ABC has been drawn, for which the base angles measure 60° and 45°. Side AB measures 2 units.



a) Determine the exact measures of the altitude and the other two sides of the triangle.

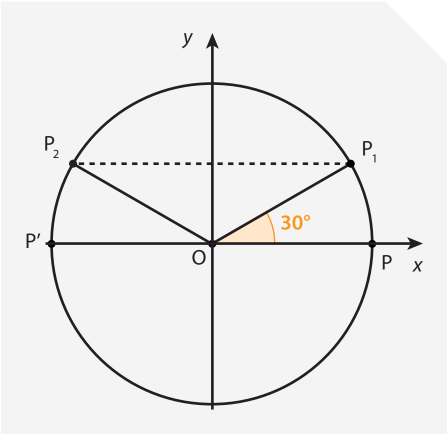
b) Deduce the exact value of sin 75° from this figure.

ANSWER KEY PAGE 294

2. Remarkable Points on the Trigonometric Unit Circle

The definition given to the sine and the cosine may apply to any angle, whether right, obtuse, straight or reflex. After determining the exact values of the sine and the cosine of the 30°, 45° and 60° angles, you can consider other points of the trigonometric unit circle. In fact, you can determine the exact values of the sine and the cosine of some of its angles by using the properties of symmetry of the circle.

8 In the trigonometric unit circle to the right, P_1 is the image of point P by a 30° rotation around centre O. Point P_2 is the symmetric point of point P_1 relative to the *y*-axis. The symmetric point P′ of point P has also been situated.



a) What angle of rotation allows point P to be applied to point P_2? Explain your reasoning.

____

____

____

____

b) What are the exact coordinates of point P_2?

____

c) In the same figure, situate the symmetric point P_3 of point P_2 relative to the *x*-axis.

What angle of rotation allows point P to be applied to point P_3? ______________________________

What are the exact coordinates of point P_3? ______________________________

d) In the same figure, situate the symmetric point P_4 of point P_3 relative to the *y*-axis.

What angle of rotation allows point P to be applied to point P_4? ______________________________

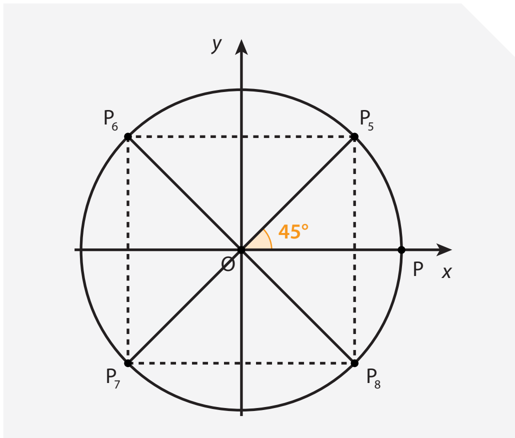
What are the exact coordinates of point P_4? ______________________________

ANSWER KEY PAGE 295

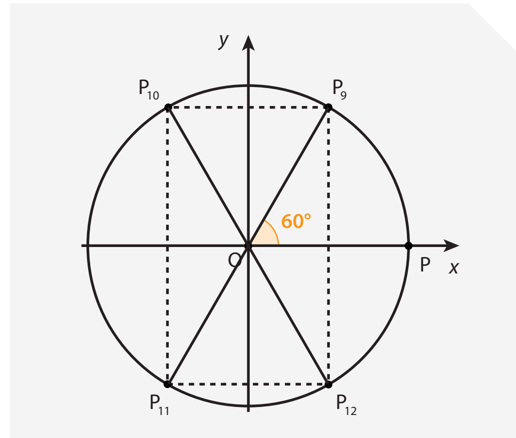
9 In the trigonometric unit circles below, the image of point P has been situated by a rotation of 45° and 60° respectively, as well as the resulting symmetric points relative to the axes.

Determine the angles of rotation associated with each of these symmetric points and their coordinates.

a)



b)



10 Complete the following table, specifying the sign of the sine and the cosine of the angle depending on the interval.

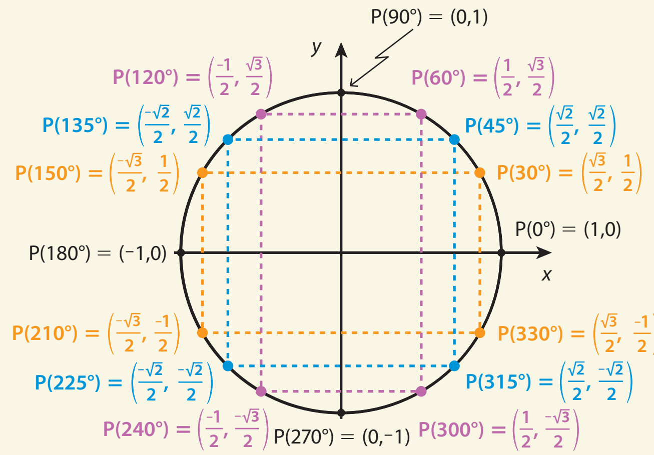
	Acute angle	Obtuse angle	Interior angle	
	$\alpha \in]0°, 90°[$	$\alpha \in]90°, 180°[$	$\alpha \in]180°, 270°[$	$\alpha \in]270°, 360°[$
Sign of cos α	+			
Sign of sin α	+			

ANSWER KEY PAGE 295

REMEMBER

Remarkable Trigonometric Points

The following representation gives the exact coordinates of 16 remarkable trigonometric points on the trigonometric unit circle.



CAUTION!

A 360° rotation of point P corresponds to one complete rotation and thus ends up at the same point as a 0° rotation. The same would apply to a rotation of 720° or −360°, for example.

In general, if α represents a given angle of rotation expressed in degrees, then

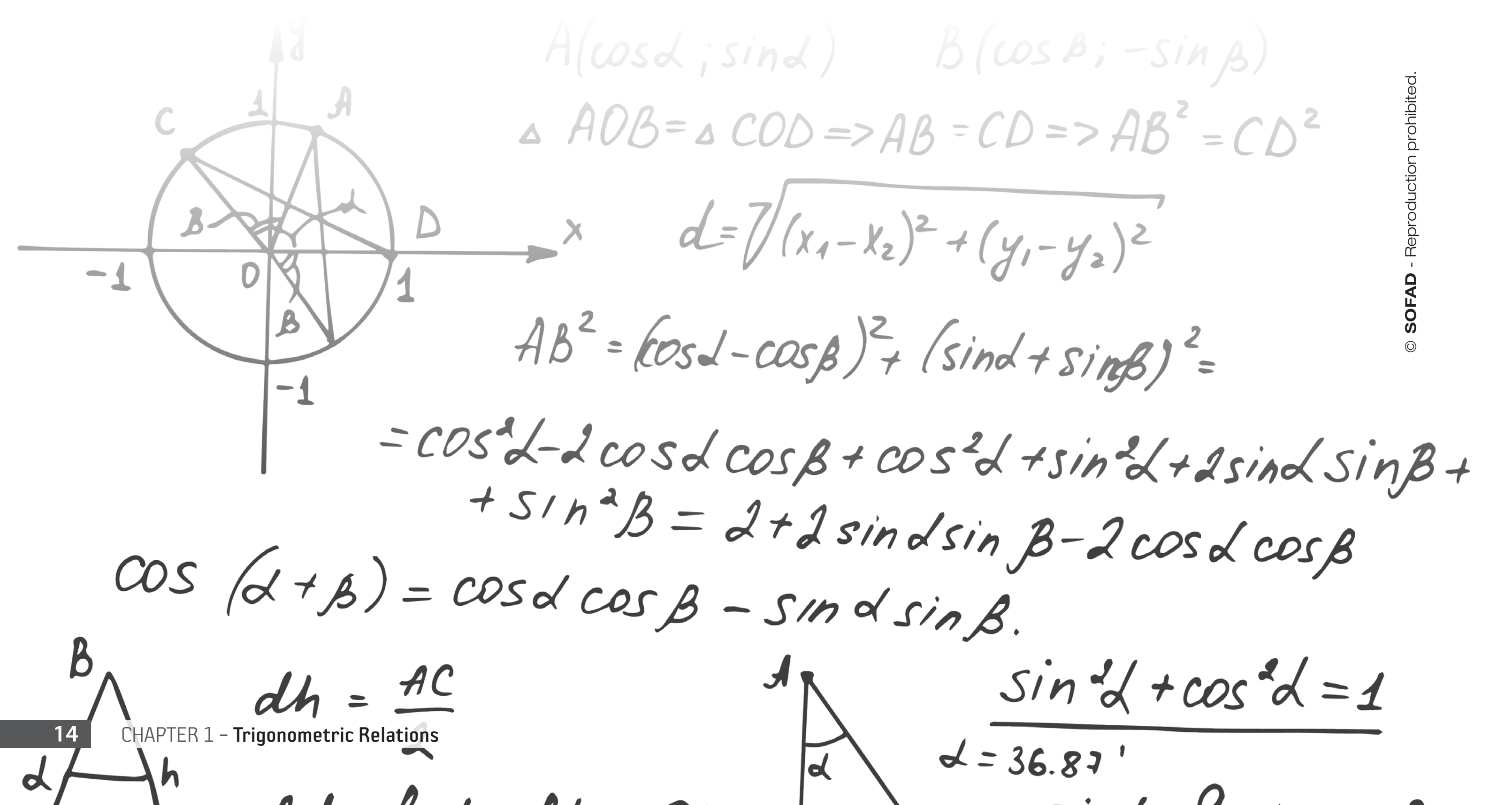
$P(\alpha) = P(\alpha \pm 360°n)$, for any integer n.

Example:

To determine the exact value of cos (−150°), you can state that
$P(-150°) = P(-150° + 360°) = P(210°)$.

The cosine of −150° thus corresponds to the x-coordinate of point P(210°).

$$\cos(-150°) = -\frac{\sqrt{3}}{2}$$



PRACTISE

11 Determine the exact values of the following sines and cosines.

a) $\sin 150° =$ ______ b) $\cos 225° =$ ______

c) $\sin(-225°) =$ ______ d) $\cos(-540°) =$ ______

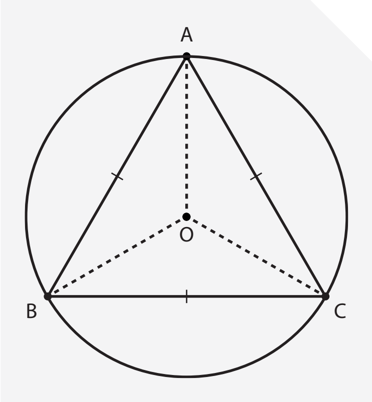
e) $\sin 420° =$ ______ f) $\cos 450° =$ ______

STRATEGY Validate with a technological tool

Even if a question asks you to determine an exact value, it may be wise to validate it with a technological tool, such as a calculator. In question 11, for example, once the $\cos 225°$ ratio is evaluated, nothing prevents you from confirming this ratio with the cos 225° key on the calculator. You then will obtain an approximation of that ratio. In the case of trigonometric ratios, rounding to the nearest ten thousandth is preferable, to be close enough to the exact value in these validation cases.

12 When an equilateral triangle is inscribed in a circle, the radii that end at each vertex form angles of 120° between them.

Determine the formula of area *A* of an equilateral triangle inscribed in a circle of radius *R*.



TIP

The following property can be useful to you.

> The area of a triangle is equal to half the sine of a reflex angle multiplied by the lengths of the adjacent sides.

Several other known properties are presented at the end of the glossary of this guide. Do not hesitate to consult them.

You now know how to interpret the sine and the cosine of any angle with the trigonometric unit circle and you can calculate the exact value of certain angles associated with remarkable trigonometric points. You have also had the opportunity to review the main trigonometric relations that allow you to determine missing measures in geometric figures. This is all you need to do to solve Situational Problem 1.1, *A Star Pendant*.

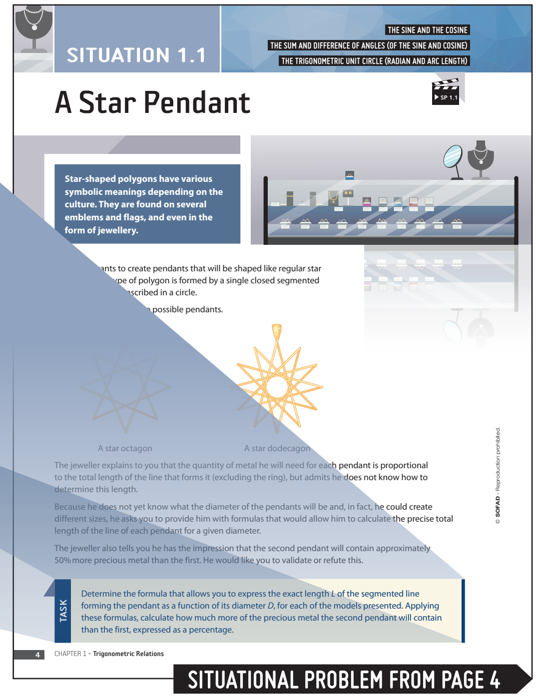
ANSWER KEY PAGE 295

SOLUTION

You are now able to complete the solution to Situational Problem 1.1.

TASK

Determine the formula that allows you to express the exact length *L* of the segmented line forming the pendant as a function of its diameter *D*, for each of the models presented. Applying these formulas, calculate how much more of precious metal the second pendant will contain than the first, expressed as a percentage.


SITUATION 1.1

THE SINE AND THE COSINE
THE SUM AND DIFFERENCE OF ANGLES (OF THE SINE AND COSINE)
THE TRIGONOMETRIC UNIT CIRCLE (RADIAN AND ARC LENGTH)

SP 1.1

A Star Pendant

Star-shaped polygons have various symbolic meanings depending on the culture. They are found on several emblems and flags, and even in the form of jewellery.

…ants to create pendants that will be shaped like regular star …ype of polygon is formed by a single closed segmented …scribed in a circle.

… possible pendants.

A star octagon

A star dodecagon

The jeweller explains to you that the quantity of metal he will need for each pendant is proportional to the total length of the line that forms it (excluding the ring), but admits he does not know how to determine this length.

Because he does not yet know what the diameter of the pendants will be and, in fact, he could create different sizes, he asks you to provide him with formulas that would allow him to calculate the precise total length of the line of each pendant for a given diameter.

The jeweller also tells you he has the impression that the second pendant will contain approximately 50% more precious metal than the first. He would like you to validate or refute this.

TASK

Determine the formula that allows you to express the exact length *L* of the segmented line forming the pendant as a function of its diameter *D*, for each of the models presented. Applying these formulas, calculate how much more of the precious metal the second pendant will contain than the first, expressed as a percentage.

4 CHAPTER 1 – Trigonometric Relations

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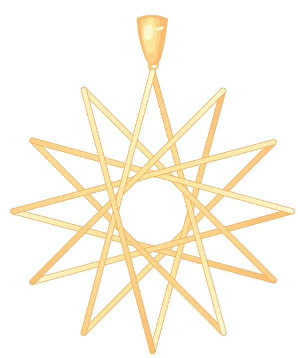
SITUATIONAL PROBLEM FROM PAGE 4

Summary of the Facts

- A jeweller wants to manufacture pendants that will be shaped like regular star polygons based on the following two models. Each pendant is formed by a single closed segmented line.

A star octagon



A star dodecagon

- The jeweller has the impression that the second pendant should contain approximately 50% more precious metal than the first. He would like you to validate or refute this.

Solution

ANSWER KEY PAGE 296

Solution (*continued*)

Answer:

Formula for star octagon: ______________________________

Formula for star dodecagon: ______________________________

Comparison of the quantity of metal in each pendant: ______________________________

____

____

ANSWER KEY PAGE 296

ACQUISITION B

Mathematical knowledge targeted:

- proving the formulas of the sine or the cosine of a sum or difference of angles
- using trigonometric relations involving the sine and the cosine of different angles
- determining arc lengths by expressing angles as radians.

1. Formulas for the Sum or Difference of Angles

The next two sections will seek to explain what the mathematicians of Antiquity and the Middle Ages did to construct their table of chords or half-chords. Transposed into the current vocabulary, this means asking yourself how you can construct a table giving the exact sine and cosine values for all multiples of a certain angle, such as all multiples of 15°. To achieve this, you must know certain formulas. For those who will continue their education at the college level, this acquisition offers the opportunity to develop an argument when demonstrating proofs.

1.1 The Sine of a Sum of Angles

You have already seen in question 7 of *Acquisition A* that it is possible to calculate the exact value of sin 75° with a certain triangle, but there is another more general method to determine this value. This is what is presented in this section. This method uses a formula that allows you to calculate the sine of a sum of two angles from the sine and the cosine of each of these angles. The formula is written as follows:

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

1 As you can see, this formula applies very well to the case of the 75° angle, which is the sum of 30° and 45°.

Complete the following calculation, which allows you to determine the exact value of sin 75°.

$\sin 75° = \sin(30° + 45°)$

$= \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_$ (Use of the formula with $\alpha = 30°$ and $\beta = 45°$)

$= \left(\frac{\square}{\square}\right) \times \left(\frac{\square}{\square}\right) + \left(\frac{\square}{\square}\right) \times \left(\frac{\square}{\square}\right)$ (Replacement of sin and cos with their value)

$= \_\_\_\_\_\_\_\_ \frac{\square}{\square} + \frac{\square}{\square}$ (Calculation of the two multiplications)

$= \frac{\square}{\square}$ (Notation of the result of the sum)

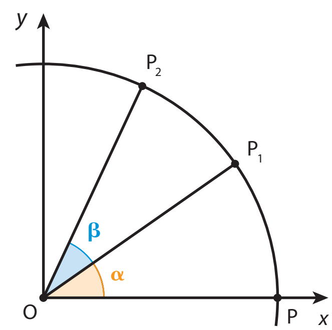
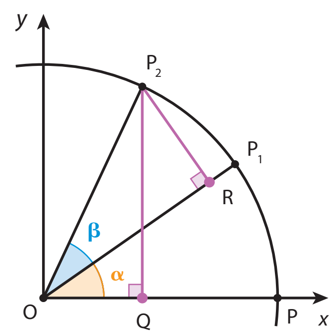
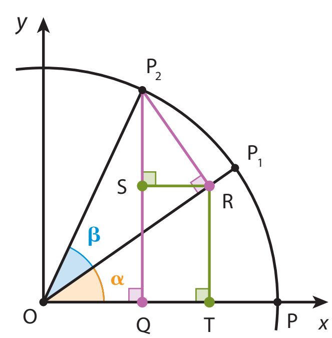
Compare the result with the one you obtained for question 7 b) of *Acquisition A*.

STRATEGY Prove a formula to better understand and retain it

It would be wise to memorize this formula allowing you to calculate the sine of a sum of two angles, because it has important applications, as you will see at a later stage of your studies. One of the best ways to memorize a formula is to understand where it comes from and, whenever possible, to prove it. This is what you will do in the next question.

ANSWER KEY PAGE 297

2 To prove the formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, you must produce the following construction.

Explanation	Construction
1) In a trigonometric unit circle, apply a rotation of an **angle α** at point P(1, 0) to obtain point P_1, and then rotate an **angle β** at point P_1 to obtain point P_2.	
2) From P_2, drop a **perpendicular P_2Q** to segment OP and a **perpendicular P_2R** to segment OP_1 to form two right triangles OP_2Q and OP_2R.	
3) To complete the figure, drop **perpendiculars RS and RT** to segments P_2Q and OP, to form rectangle RSQT.	

a) In this figure, what segment has the measure $\sin(\alpha + \beta)$? ____________________

b) Explain why you can affirm that $\sin(\alpha + \beta) = m\,\overline{RT} + m\,\overline{P_2S}$.

____

____

____

TIP

Your answer in b) shows that $\sin(\alpha + \beta)$ is the sum of the measures of two segments $\overline{RT}$ et $\overline{P_2S}$. To complete the proof, simply prove that $m\,\overline{RT} = \sin\alpha\cos\beta$ and $m\,\overline{P_2S} = \cos\alpha\sin\beta$. This is the goal of the following approach.

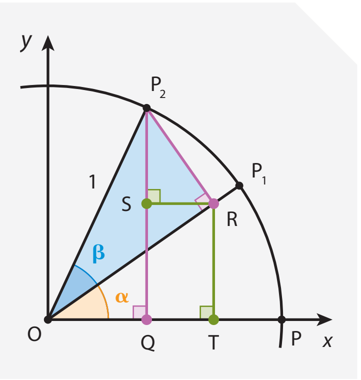
ANSWER KEY PAGE 297

c) First pay attention to right triangle OP_2R, which has a hypotenuse measuring 1 unit.

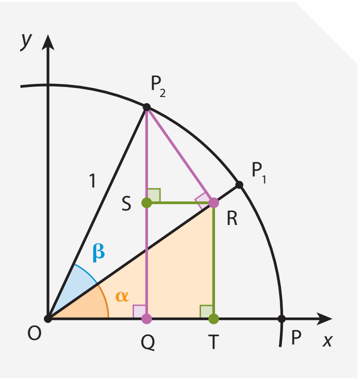
In this triangle, what trigonometric ratio is equal to:

1) the measure of side P_2R? ______________________

2) the measure of side OR? ______________________



d) Using the trigonometric ratios in right triangle ORT and one of your answers in c), prove that $m\,\overline{RT} = \sin\alpha\cos\beta$.

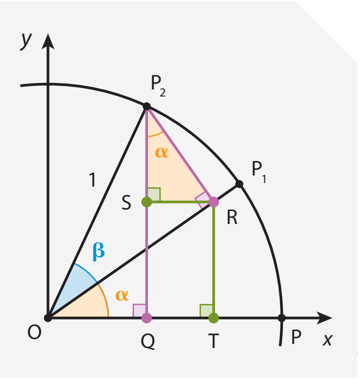


e) Now consider triangle P_2RS. Observe that the letter α has been added, indicating the measure of angle SP_2R. Explain why you can affirm that $m\angle SP_2R = \alpha$.

____

____

____



f) Using the trigonometric ratios in right triangle P_2RS and one of your answers in c), prove that $m\,\overline{P_2S} = \cos\alpha\sin\beta$.

g) Combine the statements of b), d) and f) to complete the proof.

ANSWER KEY PAGE 297

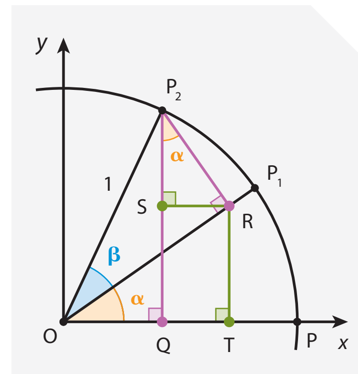
1.2 The Cosine of a Sum of Angles

Another formula can be used to calculate the cosine of a sum of angles. It is written as follows:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

To prove this formula, you will use exactly the same figure as in the previous proof.

3 To the right is the figure from the previous question once again. Remember that m $\overline{P_2R} = \sin\beta$ and m $\overline{OR} = \cos\beta$.



a) What segment in this figure has the measure $\cos(\alpha + \beta)$?

b) Explain why you can affirm that

$\cos(\alpha + \beta) = \text{m}\,\overline{OT} - \text{m}\,\overline{RS}$.

c) Prove that m $\overline{OT} = \cos\alpha\cos\beta$.

d) Prove that m $\overline{RS} = \sin\alpha\sin\beta$.

TIP

To answer question c), use right triangle ORT. To answer question d), use right triangle P_2RS.

e) Combine the statements b), c) and d) to complete the proof.

4 Using the formula of the cosine of a sum of two angles, determine the exact value of cos 75°.

$\cos 75° = \cos(30° + 45°)$

$= \_\_\_\_\_\_ - \_\_\_\_\_\_$

$= \left(\frac{\square}{\square}\right) \times \left(\frac{\square}{\square}\right) - \left(\frac{\square}{\square}\right) \times \left(\frac{\square}{\square}\right)$

$= \_\_\_\_\_\_ \frac{\square}{\square} - \frac{\square}{\square}$

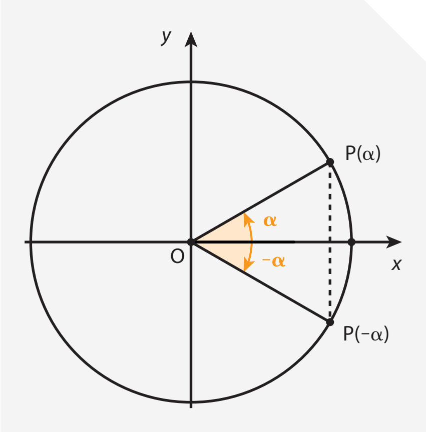
$= \frac{\square}{\square}$

ANSWER KEY PAGE 298

1.3 The Sine or the Cosine of a Difference of Angles

There are still two formulas to examine: the sine of a difference of angles and the cosine of a difference of angles. However, these formulas arise directly from the preceding formulas according to the principle that subtracting is equivalent to adding the opposite. You will deduce these formulas by first answering the following question.

5 Observe the diagram to the right in which an angle of rotation has been represented measuring α and its opposite measuring $^-\alpha$.



a) What can you state concerning $\sin(^-\alpha)$?

____

____

b) And concerning $\cos(^-\alpha)$?

____

____

6 Using the formulas for the sine and cosine of a sum, complete the following steps to deduce the difference formulas.

a) Sine of a difference of angles:

$\sin(\alpha - \beta) = \sin(\alpha + (^-\beta))$ (Subtracting is adding the opposite)

$= \sin\alpha\cos(^-\beta) + \cos\alpha\sin(^-\beta)$ (Formula of the sine of a sum of angles)

= ____________________ (Application of the properties discovered in question 5)

= ____________________ (Simplification)

b) Cosine of a difference of angles:

$\cos(\alpha - \beta) = \cos(\alpha + (^-\beta))$ (Subtracting is adding the opposite)

= ____________________ (Formula of the cosine of a sum of angles)

= ____________________ (Application of the properties discovered in question 5)

= ____________________ (Simplification)

ANSWER KEY PAGE 299

REMEMBER

Formulas for the Sine and Cosine of a Sum or Difference of Angles

For any angle α and β, you can determine the sine and the cosine of a sum or difference of angles with these four formulas.

Angle Sum Formulas

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Angle Difference Formulas

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

These formulas can serve to determine the exact value of the sine or the cosine of certain angles. You can also use them to deduce other useful formulas.

Example:

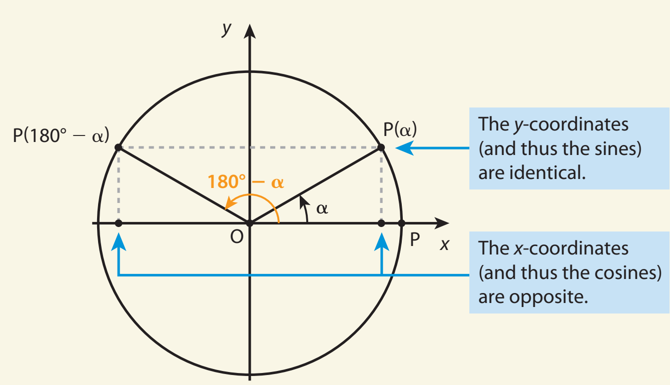
The following relations are true for any angle α.

$\sin(180° - \alpha) = \sin\alpha$	$\cos(180° - \alpha) = {}^{-}\cos\alpha$
In fact, by the formula for the sine of a difference: $\sin(180° - \alpha) = \sin 180° \cos\alpha - \cos 180° \sin\alpha$ $= (0)\cos\alpha - (^{-}1)\sin\alpha$ $= \sin\alpha$ This means that the sines of two supplementary angles are equal.	In fact, by the formula for the cosine of a difference: $\cos(180° - \alpha) = \cos 180° \cos\alpha + \sin 180° \sin\alpha$ $= (^{-}1)\cos\alpha + (0)\sin\alpha$ $= {}^{-}\cos\alpha$ This means that the cosines of two supplementary angles are the opposite of each other.

STRATEGY **Using the trigonometric unit circle to validate calculations**

The properties proved in the above example can both be deduced directly from the trigonometric unit circle with a diagram.

It is always practical to use a diagram of the trigonometric unit circle to visualize and validate calculations involving sines and cosines of angles.



PRACTISE

7 You are now able to determine the exact values of the sines and cosines of all angles that are multiples of 15°. Complete the following table of values.

Sines and Cosines of Angles that are Multiples of 15°

Angle	Sine	Cosine	Angle	Sine	Cosine
0°	0	1	105°		
15°			120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	165°		
75°			180°	0	−1
90°	1	0			

8 The formulas for the sine and cosine of the double of an angle can be deduced from the sum formulas.

a) Follow the suggested steps to discover these new formulas.

1) $\sin 2\alpha = \sin(\alpha + \alpha)$

$= \_\_\_\_\_\_\_\_\_\_\_\_$ (Formula for the sine of a sum of angles)

$= \_\_\_\_\_\_\_\_\_\_\_\_$ (Commutative property of multiplication)

$= \_\_\_\_\_\_\_\_\_\_\_\_$ (Addition of similar terms)

2) $\cos 2\alpha = \cos(\alpha + \alpha)$

$= \_\_\_\_\_\_\_\_\_\_\_\_$ (Formula of the cosine of an angle sum)

$= \_\_\_\_\_\_\_\_\_\_\_\_$ (Calculation of multiplications)

b) Show that the last formula you deduced in a) can also be written in the following form.

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

c) Validate the three formulas you used to calculate the sine and the cosine of 60°, given that 60° is double 30°.

ANSWER KEY PAGE 299

2. Application of Trigonometric Relations

You now know several trigonometric relations that allow you to establish connections between the sines and cosines of certain angles. The following are some other ways to deduce the sine or the cosine of various angles.

9 Complete the equations describing each of the main trigonometric relations.

a) Property of complementary angles: $\sin(90° - \alpha) =$ ________ $\cos(90° - \alpha) =$ ________

b) Property of supplementary angles: $\sin(180° - \alpha) =$ ________ $\cos(180° - \alpha) =$ ________

c) **Identity** resulting from the Pythagorean theorem: ________ + ________ = 1

By isolating $\sin\alpha$ or $\cos\alpha$, the identity can also be written in one of the following forms:

$\sin\alpha =$ ________ $\cos\alpha =$ ________

d) The sum formulas: $\sin(\alpha + \beta) =$ ________

$\cos(\alpha + \beta) =$ ________

e) The difference formulas: $\sin(\alpha - \beta) =$ ________

$\cos(\alpha - \beta) =$ ________

REMEMBER

Application of Trigonometric Relations

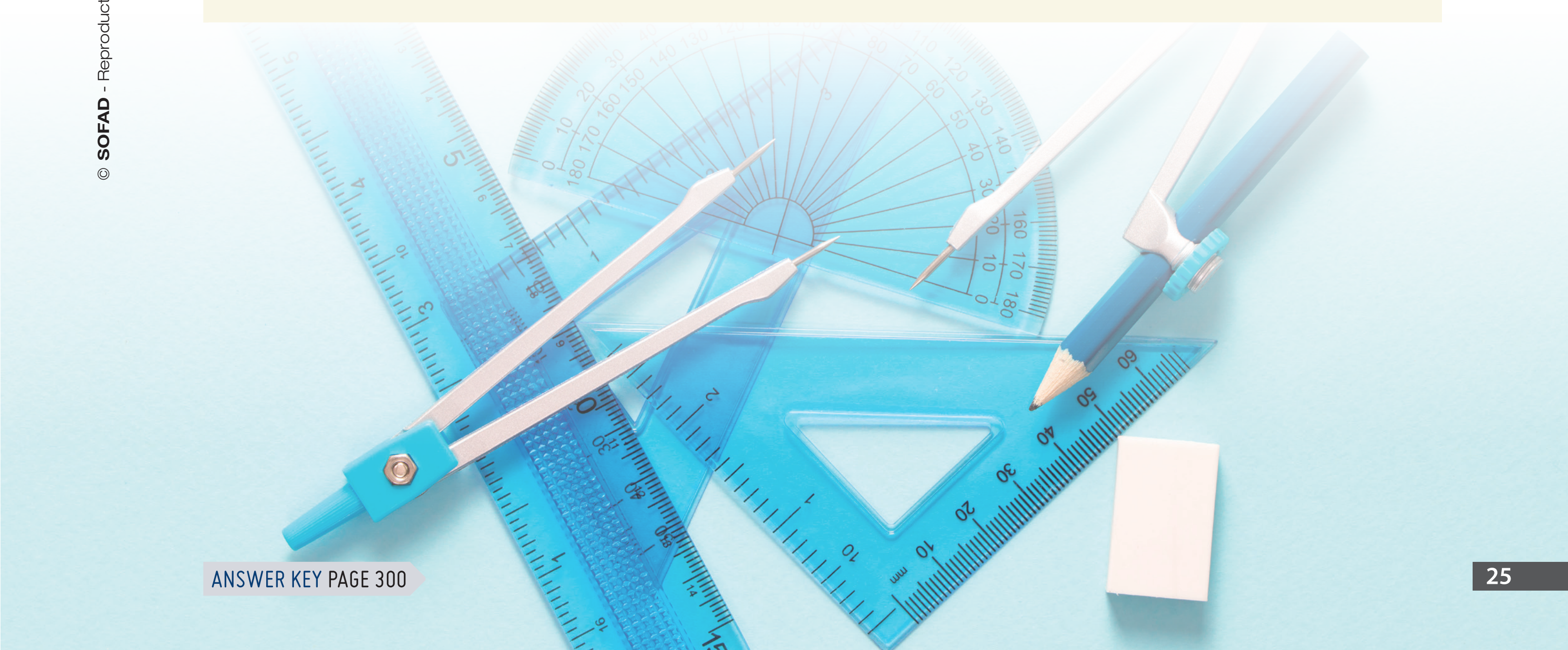
Knowing the measure of a sine or a cosine of an angle, or an approximation of this measure, it is possible to use trigonometric relations to determine the sine or the cosine of different angles related to it.

Example:

Given that $\cos 36° = \frac{1 + \sqrt{5}}{4} \approx 0.8090$, it is possible to deduce the sine and the cosine of other angles, such as: $\sin 66°$, $\cos 24°$ or $\cos 156°$.

TIP

In question 10 of *Integration*, you will learn to use the properties of the regular pentagon to determine the exact value of cos 36°.



ANSWER KEY PAGE 300

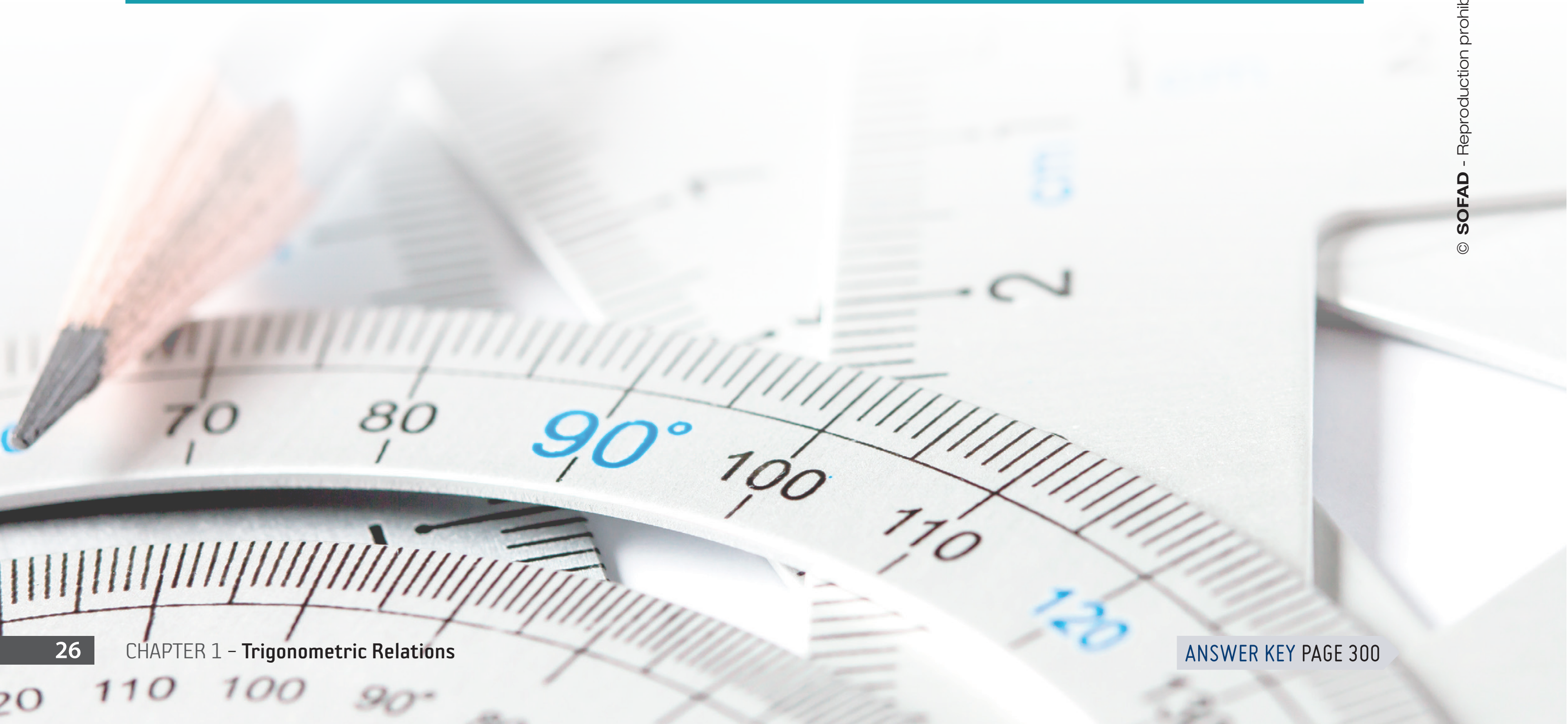
Deduced Values	Justifications
1) $\sin 36° = \sqrt{1 - \cos^2 36°}$ $\approx \sqrt{1 - 0.8090^2}$ ≈ 0.5878	By the identity $\sin^2\alpha + \cos^2\alpha = 1$, which can be written as $\sin\alpha = \pm\sqrt{1 - \cos^2\alpha}$, and by the fact that P(36°) is situated in the first quadrant (so sin 36° is positive).
2) $\sin 66° = \sin(36° + 30°)$ $= \sin 36° \cos 30° + \cos 36° \sin 30°$ $\approx 0.5878 \times \frac{\sqrt{3}}{2} + 0.8090 \times \frac{1}{2}$ ≈ 0.9135	By the formula for the sine of an angle sum: $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
3) $\cos 24° = \sin(90° - 24°)$ $= \sin 66°$ ≈ 0.9135	By the property of complementary angles: $\cos\alpha = \sin(90° - \alpha)$
4) $\cos 156° = {}^-\cos(180° - 24°)$ $= {}^-\cos 24°$ $\approx {}^-0.9135$	By the property of supplementary angles: $\cos(180° - \alpha) = {}^-\cos\alpha$

PRACTISE

10 Given that $\cos 75° \approx 0.2588$ and $\cos 36° \approx 0.8090$, approximate the value of $\cos 3°$.

NOTE!

This first chapter concerning the concepts involved in trigonometric relations and proofs is the most difficult of the MTH-5173-2 course. You can catch your breath in the next few chapters. Keep up the effort!



ANSWER KEY PAGE 300

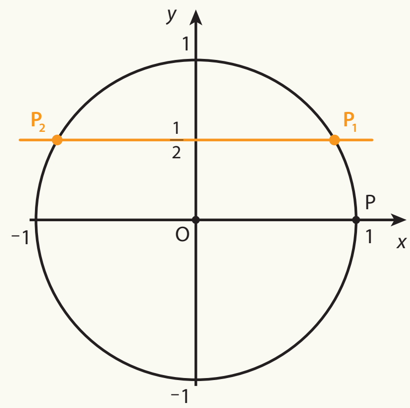
3. Looking for the Measure of an Angle

Another type of question still has to be examined: how to determine the measure of an angle for which the sine or the cosine is known. You should already understand that an infinite number of solutions exist if you consider all the possible angles of rotation. However, the inverse operations **arc sine** and **arc cosine** found on calculators only give one solution. All the other possible solutions must be deduced. Before addressing the problem raised by using a calculator, you will examine a situation in which all the solutions can be determined by geometric reasoning.

11 For an angle of rotation α, such as $\sin\alpha = \frac{1}{2}$.

a) What is the possible measure of this angle?

NOTE: By definition, $\sin\alpha$ is the *y*-coordinate of trigonometric point $P(\alpha)$. As you can see in the representation to the right, the line of equation $y = \frac{1}{2}$ crosses the trigonometric unit circle at two points: P_1 and P_2.



1) Draw the angles of rotation α_1 and α_2 between 0° and 360° associated with each of these points.

2) Given what you know about the remarkable trigonometric points, determine the measure of these angles. ____________________

b) What relation can you establish between the measures of these angles?

____

c) Because the angles sought are angles of rotation, they might be greater than 360° or they might even be negative. How would you describe all the possible values that angle α can have?

____

12 Now examine a more general situation, in which the possible situations are not linked to remarkable trigonometric points. The same question is asked with one slight change.

a) For an angle of rotation α, such as $\sin\alpha = -0.6$, what is the possible measure of this angle?

1) Draw a diagram with a trigonometric unit circle as in the previous question, approximately situating the position of the line of the equation $y = -0.6$.

2) In which quadrant do you find the intersection points of the circle and this line?

____

ANSWER KEY PAGE 300

b) With a calculator, determine the value of $\sin^{-1}(-0.6)$ to the nearest tenth of a degree. Then draw the corresponding angle in your previous diagram.

$\sin^{-1}(-0.6) \approx$ ____________________

c) Determine the measure of the angle associated with another intersection point in your diagram.

____

d) Describe all the possible measures the angle of rotation sought could have.

____

> **CAUTION!**
>
> The inverse sine and cosine operations, called arc sine and arc cosine, allow you to obtain a possible measure of an angle for which you know the sine or the cosine. Often they are represented by the expressions $\sin^{-1}$ and $\cos^{-1}$.
>
> On most calculators, these operations are accessed by the sin and cos keys, after first pressing the 2ND key.

13 You could follow a similar approach with the cosine. This time suppose you are looking for an angle of rotation for which the cosine is equal to $-\frac{3}{4}$. This means solving the equation $\cos\alpha = -\frac{3}{4}$.

a) Draw a diagram that represents the situation.

> **CAUTION!**
>
> You must account for the fact that $\cos\alpha$ is the *x*-coordinate of point $P(\alpha)$. You are looking for points for which the *x*-coordinate (and not the *y*-coordinate) is equal to $-\frac{3}{4}$.

b) With a calculator, determine the value of $\cos^{-1}\left(-\frac{3}{4}\right)$ to the nearest tenth of a degree. ____________________

c) Determine the angle measure associated with the symmetric intersection point in your diagram. Then describe all the possible measures the angle of rotation sought could have.

____

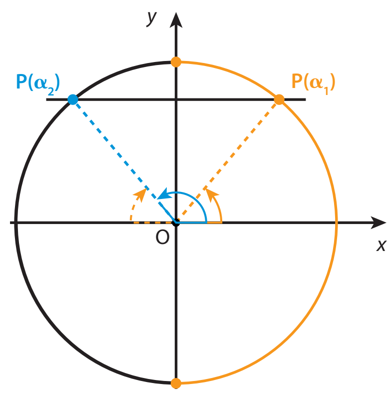
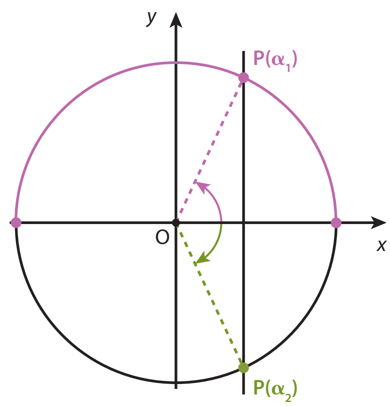
ANSWER KEY PAGE 301

REMEMBER

Determining the Measure of an Angle for Which You Know the Sine or the Cosine

To determine the measure of an angle of rotation for which you know the sine or the cosine, you can proceed as follows:

- determine a first solution α_1 by using the inverse operation arc sin (or $\sin^{-1}$) or arc cos (or $\cos^{-1}$) of the calculator, as needed
- determine (if applicable) a second solution α_2, symmetrical to the first in the trigonometric unit circle
- deduce all the other relevant solutions by adding or subtracting multiples of 360° to or from solutions α_1 and α_2.

If you know the sine of the angle	If you know the cosine of the angle
	
The result of the inverse operation arc sin is always an angle situated in the interval [−90°, 90°].	The result of the inverse operation arc cos is always an angle situated in the interval [0°, 180°].
The symmetric solution is given by the relation:	The symmetric solution is given by the relation:
$\alpha_2 = 180° - \alpha_1$	$\alpha_2 = {}^{-}\alpha_1$

PRACTISE

14 Among all the angles of rotation situated between 360° and 720°, which ones have a cosine equal to $\frac{\sqrt{3}}{2}$?

15 A triangle inscribed in a circle 10 cm in diameter has a side 8 cm long. What is the possible measure of the reflex angle of the triangle opposite this side?

TIP

When a triangle ABC is inscribed in a circle, the proportion $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, which arises from the sine law, has a special meaning. The three ratios of this proportion are all equal to the diameter of the circle.

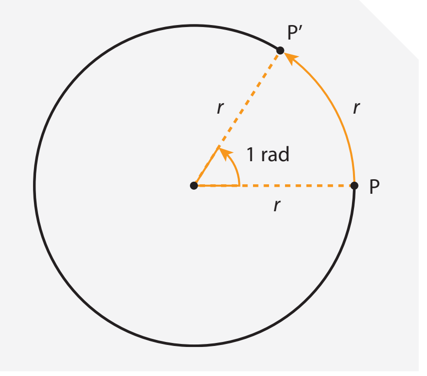
ANSWER KEY PAGE 301

4. Measuring Angles in Radians

So far, all angles have been measured in degrees, because this is the procedure generally followed in geometry. However, there is another field of application of trigonometry, the field of algebraic functions, where a different procedure is followed. In the course MTH-5171 – *Algebraic and Graphical Modelling in a Fundamental Context 2*, when addressing the sine function, for which the rule is $f(x) = \sin x$, the independent variable x may represent many things other than an angle. This may be an arc length, but most often it will be time elapsed or simply a unitless number. To define this function logically, the concept of **radian** is used. As you will discover, this unit of angle measure also has its use in geometry.

16 The following is a definition of the radian.

> A point on a circle rotates by 1 radian (notated rad) if the arc it travels by turning in the positive direction has the same length as the radius of the circle.



You must understand that the angle of rotation is 2 rad if the length of the arc travelled is equal to twice the radius; similarly, the angle of rotation is 3 rad if the arc length is equal to three times the radius, etc.

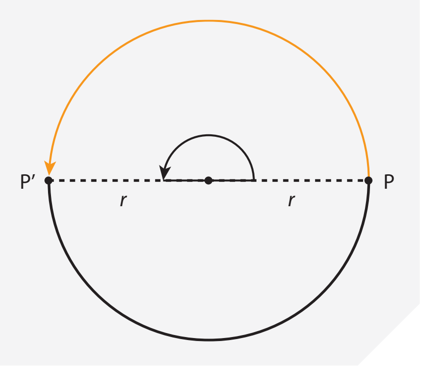
a) Let r be the radius of the circle. Point P is rotated one half-turn counterclockwise, for a rotation of 180°.

1) What expression represents the length of the arc travelled?

____

2) Deduce the exact measure of the angle of rotation in radians.

____



b) Based on your previous answer, determine the exact value in radians of the following angles.

30°: ________ 45°: ________ 60°: ________ 90°: ________

120°: ________ 135°: ________ 150°: ________ 270°: ________

c) What is the value in degrees of the following angles of rotation?

$\frac{5\pi}{4}$ rad: ________ $\frac{5\pi}{3}$ rad: ________ $\frac{11\pi}{6}$ rad: ________ 2π rad: ________

d) Determine to the nearest tenth the measure in degrees of an angle of 1 rad.

ANSWER KEY PAGE 301

17 On a circular track with a radius of 24 m, a person runs counterclockwise at a speed of 4 m/s.

a) What arc length will this person have travelled after 30 s? ____________________

b) What angle of rotation (in degrees or in radians) can be associated with this displacement?

c) Another runner has covered $\frac{3}{4}$ of the track in the same time. What angle of rotation can be associated with this runner's movement? What distance has he travelled?

REMEMBER

Angle Measure in Radians and Arc Length

A point on a circle is rotated 1 radian (notated as rad), if the arc it travels by turning in the positive direction has the same length as the radius of the circle.

By comparing the two angle measures, in degrees and in radians, associated with one half-turn, you can establish the following equivalence:

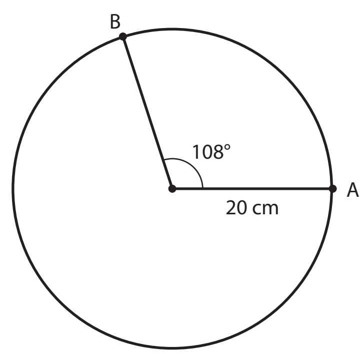
$$180° = \pi \text{ rad}$$

A simple relation exists between length L of an arc in a circle of radius r and measure θ in radians of the centre angle that intercepts it.

$$L = r\theta$$

Example:

In a circle with a radius of 20 cm, centre angle AOB measures 108°.



You can determine the measure of the angle in radians by proportional reasoning.

$$\frac{108°}{180°} = \frac{\theta \text{ rad}}{\pi \text{ rad}} \quad \Rightarrow \quad \theta = \frac{108\pi}{180} = \frac{3\pi}{5}$$

The angle measures $\frac{3\pi}{5}$ rad.

This easily allows you to deduce the length of arc AB.

$$m\,\widehat{AB} = 20 \times \frac{3\pi}{5} = 12\pi \approx 37.7 \text{ cm}$$

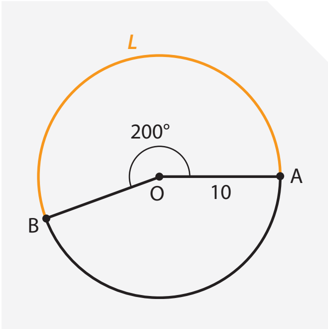
The arc has a length of approximately 37.7 cm.

ANSWER KEY PAGE 302

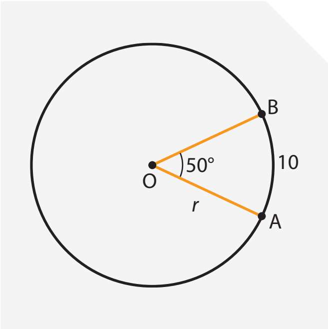
PRACTISE

18 Determine the requested measure. If applicable, round to the nearest tenth.

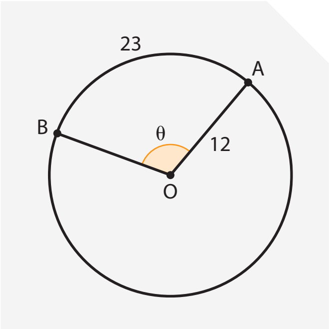
a) The measure of the large arc AB



b) The radius of the circle



c) The measure of the angle θ (in degrees)



19 A train wheel is composed of two disks of different sizes. The part that is supported on the rails has a diameter of 1.25 m while the part that runs within the rails has a diameter of 1.4 m.

a) What angle of rotation, in degrees, does the wheel make around its centre if the train travels 100 m?

b) During this movement, a point situated on the contour of the larger disk travelled a certain arc length relative to the centre of the wheel. What is this length?

ANSWER KEY PAGE 302

CONSOLIDATION

1 Determine the exact value of the following trigonometric ratios.

a) sin 60° ______________________ b) cos (⁻30°) ______________________

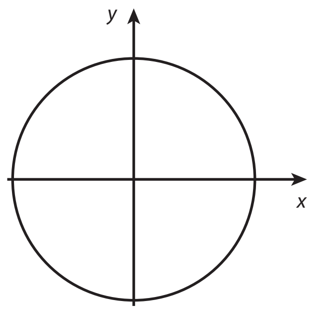
c) cos (⁻225°) ______________________ d) sin 210° ______________________

e) sin 495° ______________________ f) cos (⁻540°) ______________________

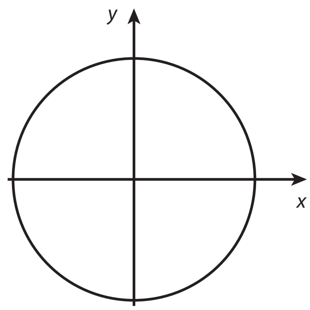
g) sin (⁻810°) ______________________ h) cos 1140° ______________________

2 Using the trigonometric unit circle, determine the measures of each angle between 0° and 360° that satisfy the following equations. Round the angle measures to the nearest tenth.

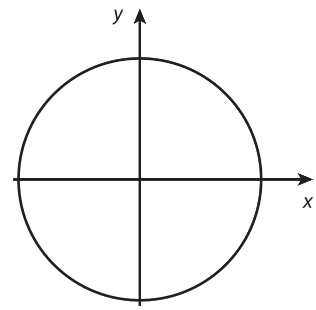
a) $\sin \alpha = -\frac{1}{2}$



b) $\cos \beta = \frac{\sqrt{3}}{2}$



c) $\sin \theta = \frac{\sqrt{2}}{2}$



3 Determine to the nearest tenth of a degree the measure of all the angles contained in the given intervals that satisfy the following equations.

a) $\cos \alpha = -0.7$, where $\alpha \in [180°, 360°]$

b) $\sin \beta = 0.2$, where $\beta \in [-90°, 0°]$

c) $\cos \theta = 0.9$, where $\theta \in [540°, 900°]$

d) $\sin \varphi = -0.72$, where $\varphi \in \,]990°, 1710°[$

DID YOU KNOW?

The word sine used in mathematics comes from a translation error. In fact, the term *jyâ*, used by the mathematician-astronomer Aryabhata in the 6th century to designate the length of half-chords, was adopted by the Arabs in the form *jîba*. Later in the Middle Ages, those who translated the Arabic texts into Latin confused *jîba* with another similar Arabic word, *ja-îb*, which means a pocket or a fold in a garment, and which is translated in Latin as *sinus*.

ANSWER KEY PAGE 303

4 Given that $\sin \alpha = 0.6$ and $\cos \beta = 0.28$, and that α and β are acute angles, determine the values of the following trigonometric expressions.

a) $\cos \alpha$

b) $\sin \beta$

c) $\sin (\alpha - \beta)$

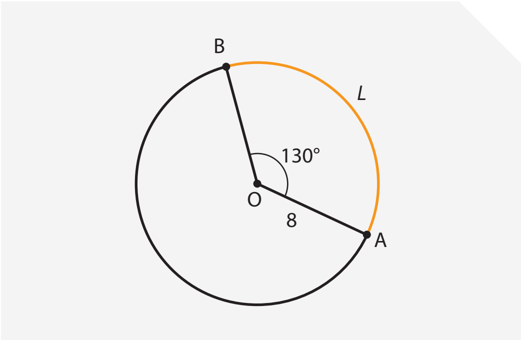
d) $\cos (\alpha - \beta)$

e) $\sin (\alpha + \beta)$

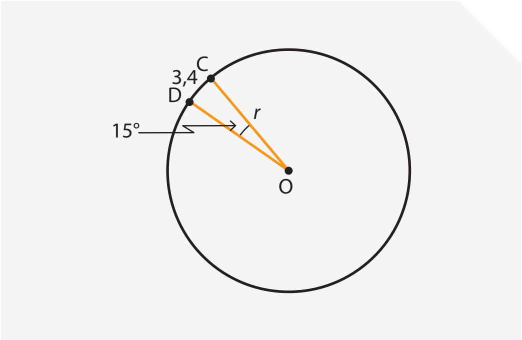
f) $\cos (\alpha + \beta)$

5 Determine the requested measure. If applicable, round to the nearest hundredth.

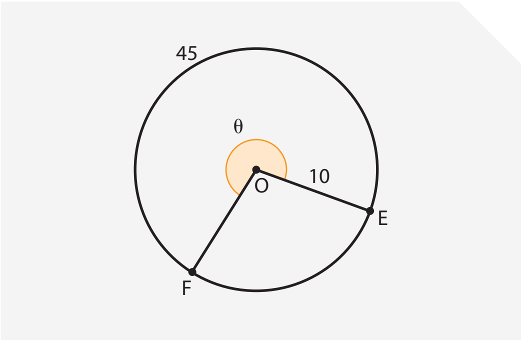
a) The measure of the small arc AB



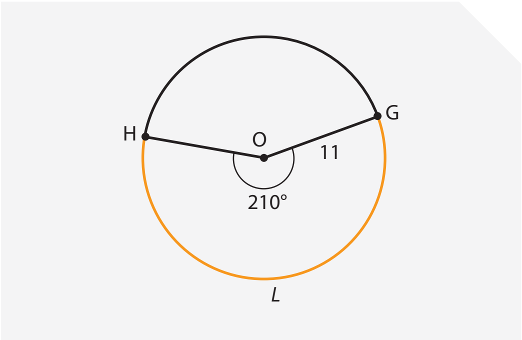
b) The radius of the circle



c) The measure of angle θ (in degrees)

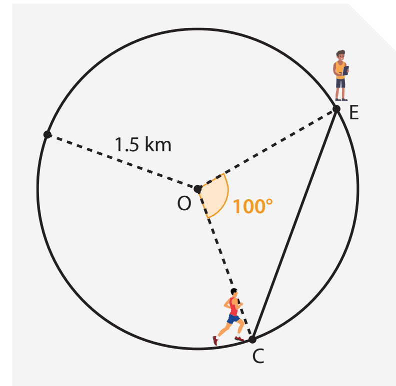


d) The measure of the large arc GH



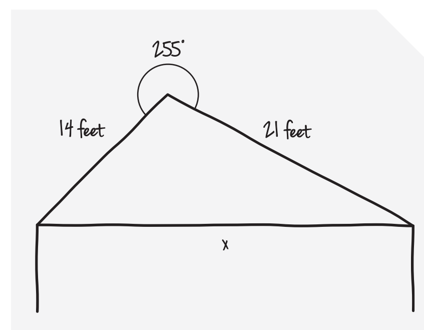
ANSWER KEY PAGE 303

6 A runner on a circular training track wants to get to his coach. Instead of following the running trail, he decides to take the shortest route. Considering that the radius of the running trail is 1.5 km and that the angle formed by the coach and the runner relative to the centre of the trail is 100°, determine the number of km the runner saved by taking his shortcut instead of following the trail.



7 Philippe wants to make a scale plan of his house. Since it is squeezed between two other houses, it is impossible for him to measure its depth precisely with a measuring tape. He thinks about another way to proceed and decides to climb on his roof. He measures the two pitches of his roof in feet and makes a sketch of the side view of his house.

Determine the depth of Philippe's house as precisely as possible to the nearest hundredth of a foot.



TIP

In this type of plan, it is essential to use measurements that are as precise as possible. Unfortunately, a mathematical solution often involves several roundings that stray farther and farther from the value sought at each step. It is then mandatory to work with the exact values, only rounding the final result.

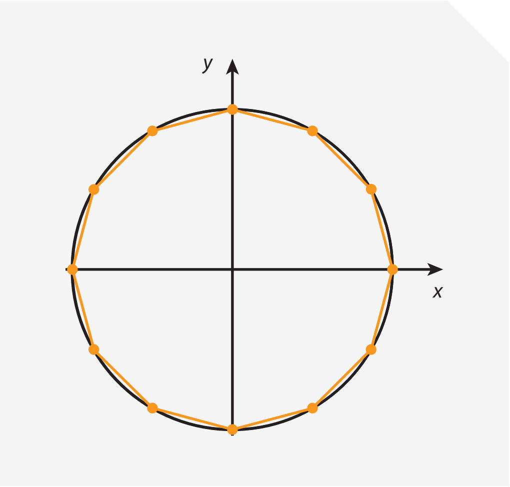
DID YOU KNOW?

In Québec, the construction field still uses the units of measure of the Imperial system, even though the use of the metric system is required in the Weights and Measures Act.

ANSWER KEY PAGE 304

8 Below is a regular dodecagon inscribed in a circle.

a) Prove that the area of a regular dodecagon inscribed in a circle of radius r is given by the formula $A = 3r^2$.

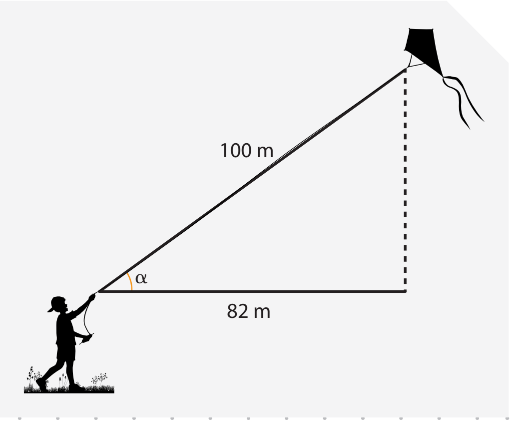


b) Deduce a formula giving the area of the regular dodecagon as a function of the measure of its side c.

9 A child is holding a kite at the end of a 100 m string that will be assumed to be stretched enough to be represented by a line segment. The horizontal distance between the kite and the child then is 82 m, as shown in the illustration to the right.

A few minutes later, the child finds that the angle α the kite initially formed with the horizontal has doubled, while the stretched string still measures 100 m.

By what percentage has the height of the kite increased?



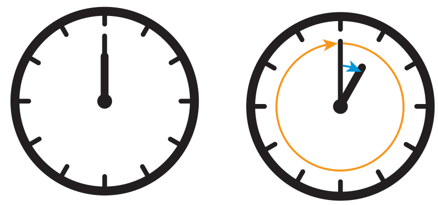
ANSWER KEY PAGE 305

10 The little hand of a clock measures 2 cm and the big hand measures 3 cm.

Between midnight and 1 a.m., at what times, to the nearest minute, is the distance between the tips of the two hands 4 cm?

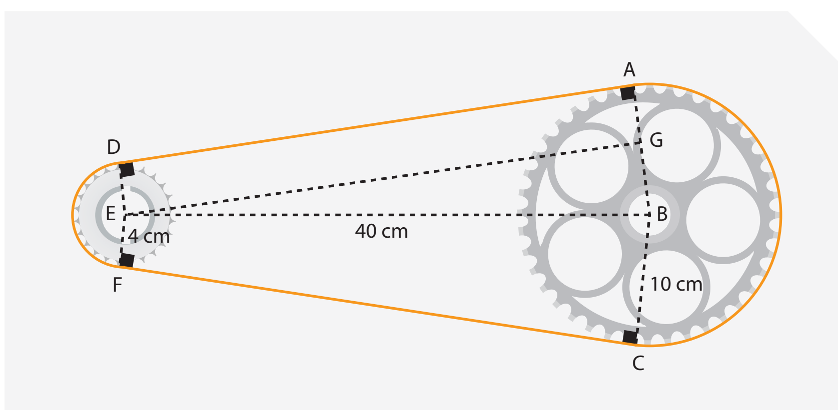
TIP

You could use the fact that in 1 h, the big hand makes one complete clockwise rotation, or a rotation of $^{-}360°$, while the little hand only makes one twelfth of a rotation in the same direction, a rotation of $^{-}30°$. The difference between the rotation speeds of the two hands is therefore $^{-}330°$/h.



11 A chain connects the pinion (10 cm radius) of a bicycle pedal to the pinion (4 cm radius) of the rear wheel. The distance between the centres of the two pinions is 40 cm.

In the diagram below, the chain is represented by the orange line. In addition, a line has been drawn parallel to side AD, passing through E to show a right triangle.



a) Determine the measure of angles ABE and DEB in radians. Round to the nearest hundredth.

b) Determine the length of the chain to the nearest centimetre.

ANSWER KEY PAGE 306

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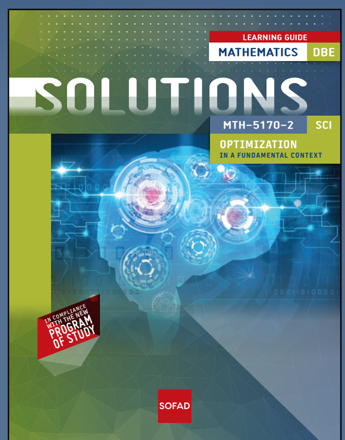

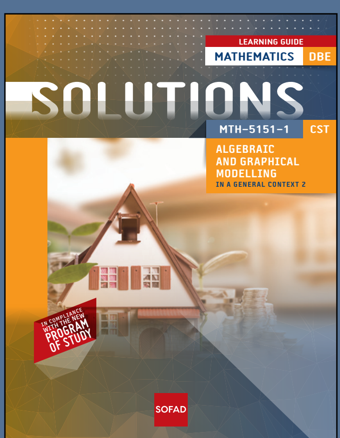

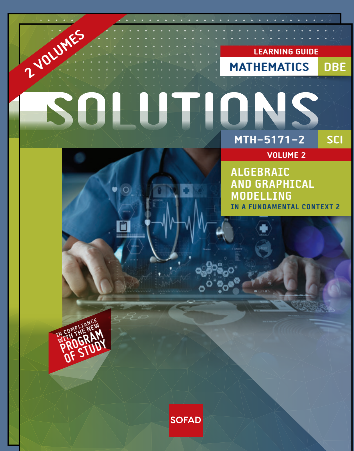

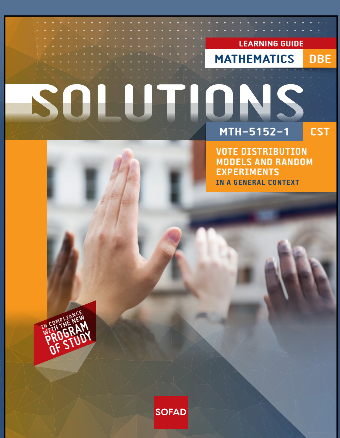

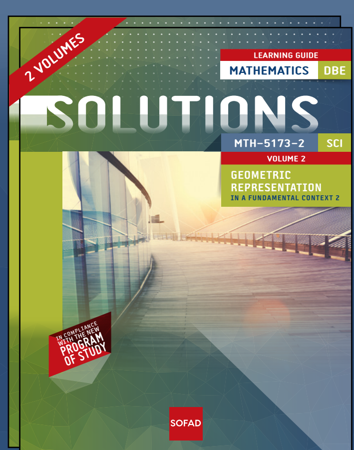

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